The Renormalization Scale Problem

- No renormalization scale ambiguity in QED
- Running Gell Mann-Low-Dyson QED Coupling sums all Vacuum Polarization Contributions
- QED Scale Identical to Photon Virtuality
- Examples: Lamb Shift, muonic atoms, g-2
- No renormalization scale ambiguity in EW theory
- Dressed Skeleton Expansion

$$\rho = C_0 \alpha_s(Q) \left[1 + C_1(Q) \frac{\alpha_s(Q)}{\pi} + C_2(Q) \frac{\alpha_s^2(Q)}{\pi^2} + \cdots \right].$$

How does one set scale Q?

Electron-Electron Scattering

$$\mathcal{M}_{ee\to ee}(++;++) = \frac{8\pi s}{t}\,\alpha(t) + \frac{8\pi s}{u}\,\alpha(u)$$

- No scale ambiguity! Two separate scales. Gauge Invariant. Dressed photon propagator
- This choice sums all vacuum polarization, nonzero beta terms into running coupling.
- If one chooses a different scale, one must sum an infinite number of graphs -- but then recover same result!
- Number of active leptons correctly set -- all t, u.

$$M(e^+e^- \to e^+e^-) \propto \alpha(s)$$

Has correct analytic / unitarity thresholds for ImM at $s = 4m^2_{\ell^+\ell^-}$

No other scale correct. If one chooses another scale, e.g.,

$$\mu_R^2 = 0.9s,$$

then must resum infinite number of vacuum polarization diagrams.

Recover $\alpha(s)$.

QCD Lagrangian

gluon dynamics

quark kinetic energy + quark-gluon dynamics

mass term

$$\boxed{L_{QCD} = -\frac{1}{4g^2}\,\mathrm{Tr}(G^{\mu\nu}G_{\mu\nu}) + \sum_{f=1}^{nf} i\,\overline{\psi_f}\,D_\mu\,\gamma^\mu\,\psi_f + \sum_{f=1}^{nf} m_f\,\overline{\psi_f}\,\psi_f}$$

QCD color charge

field strength tensor

covariant derivative

quark field

$\lim N_C \to 0$ at fixed $\alpha = C_F \alpha_s, n_\ell = n_F / C_F$

QCD $\to$ Abelian Gauge Theory

Huet, sjb

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On the elimination of scale ambiguities in perturbative quantum chromodynamics

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We present a new method for resolving the scheme-scale ambiguity that has plagued perturbative analyses in quantum chromodynamics (QCD) and other gauge theories. For Abelian theories the method reduces to the standard criterion that only vacuum-polarization insertions contribute to the effective coupling constant. Given a scheme, our procedure automatically determines the coupling-constant scale appropriate to a particular process. This leads to a new criterion for the convergence of perturbative expansions in QCD. We examine a number of well known reactions in QCD, and find that perturbation theory converges well for all processes other than the gluonic width of the Υ. Our analysis calls into question recent determinations of the QCD coupling constant based upon Υ decay.

BLM Scale Setting

BLM Scale Setting

$$\rho = C_0 \alpha_{\overline{MS}}(Q) \left[1 + \frac{\alpha_{\overline{MS}}(Q)}{\pi} \left(-\tfrac{3}{2}\beta_0 A_{VP} + \tfrac{33}{2} A_{VP} + B \right) + \cdots \right]$$

by

$$\rho = C_0 \alpha_{\overline{MS}}(Q^*) \left[1 + \frac{\alpha_{\overline{MS}}(Q^*)}{\pi} C_1^* + \cdots \right],$$

where

$$Q^* = Q \exp(3A_{VP}),$$

$$C_1^* = \tfrac{33}{2} A_{VP} + B .$$

The term $33A_{VP}/2$ in C_1^* serves to remove that part of the constant B which renormalizes the leading-order coupling. The ratio of these gluonic corrections to the light-quark corrections is fixed by $\beta_0 = 11 - \frac{2}{3} n_f$.

Use n_f dependence at NLO to identify A_{VP}

Skeleton expansion: Gardi, Rathsman, sjb

Features of BLM Scale Setting

- All terms associated with nonzero beta function summed into running coupling
- BLM Scale Q* sets the number of active flavors
- Only n_f dependence required to determine renormalization scale at NLO
- Result is scheme independent: Q* has exactly the correct dependence to compensate for change of scheme
- Correct Abelian limit

Features of BLM Scale Setting

- All terms associated with nonzero beta function summed into running coupling
- Resulting series identical to conformal series
- Renormalon n! growth of PQCD coefficients from beta function eliminated!
- In general, BLM scale depends on all invariants

$$R_{e^+e^-}(Q^2) \equiv 3 \sum_{\text{flavors}} e_q{}^2 \left[1 + \frac{\alpha_R(Q)}{\pi} \right].$$

$$R_{e^+e^-}(Q^2) = 3 \sum_q e_q{}^2 \left[1 + \frac{\alpha_{\overline{\rm MS}}(Q)}{\pi} + \frac{\alpha_{\overline{\rm MS}}{}^2}{\pi^2}(1.98 - 0.115 n_f) + \cdots \right]$$

$$\to 3 \sum_q e_q{}^2 \left[1 + \frac{\alpha_{\overline{\rm MS}}(Q^*)}{\pi} + \frac{\alpha_{\overline{\rm MS}}{}^2(Q^*)}{\pi^2} 0.08 + \cdots \right],$$

$\boxed{Q^* = 0.710Q.}$ Notice that $\alpha_R(Q)$ differs from $\alpha_{\overline{\rm MS}}(Q^*)$ by only $0.08\alpha_{\overline{\rm MS}}/\pi$, so that $\alpha_R(Q)$ and $\alpha_{\overline{\rm MS}}(0.71Q)$ are effectively interchangeable (for any value of n_f).

BLM Scale Setting

Deep-inelastic scattering. The moments of the nonsinglet structure function $F_2(x,Q^2)$ obey the evolution equation

$$
\begin{aligned}
Q^2\frac{d}{dQ^2}&\ln M_n(Q^2)\\
&= -\frac{\gamma_n^{(0)}}{8\pi}\alpha_{\overline{\rm MS}}(Q)\left[1+\frac{\alpha_{\overline{\rm MS}}}{4\pi}\frac{2\beta_0\beta_n+\gamma_n^{(1)}}{\gamma_n^{(0)}}+\cdots\right]\\
&\to -\frac{\gamma_n^{(0)}}{8\pi}\alpha_{\overline{\rm MS}}(Q_n^*)\left[1-\frac{\alpha_{\overline{\rm MS}}(Q_n^*)}{\pi}C_n+\cdots\right],
\end{aligned}
$$

where, for example,

$$Q_2^* = 0.48Q, \quad C_2 = 0.27\,,$$

$$Q_{10}^* = 0.21Q, \quad C_{10} = 1.1\,.$$

For n very large, the effective scale here becomes $Q_n^* \sim Q/\sqrt{n}$

BLM scales for DIS moments

$$V(Q^2) = -\frac{C_F 4\pi \alpha_{\overline{\rm MS}}(Q)}{Q^2}\left[1 + \frac{\alpha_{\overline{\rm MS}}}{\pi}\left(\tfrac{5}{12}\beta_0 - 2\right) + \cdots\right] \tag{19a}$$

$$\to -\frac{C_F 4\pi \alpha_{\overline{\rm MS}}(Q^*)}{Q^2}\left[1 - \frac{\alpha_{\overline{\rm MS}}(Q^*)}{\pi} 2 + \cdots\right], \tag{19b}$$

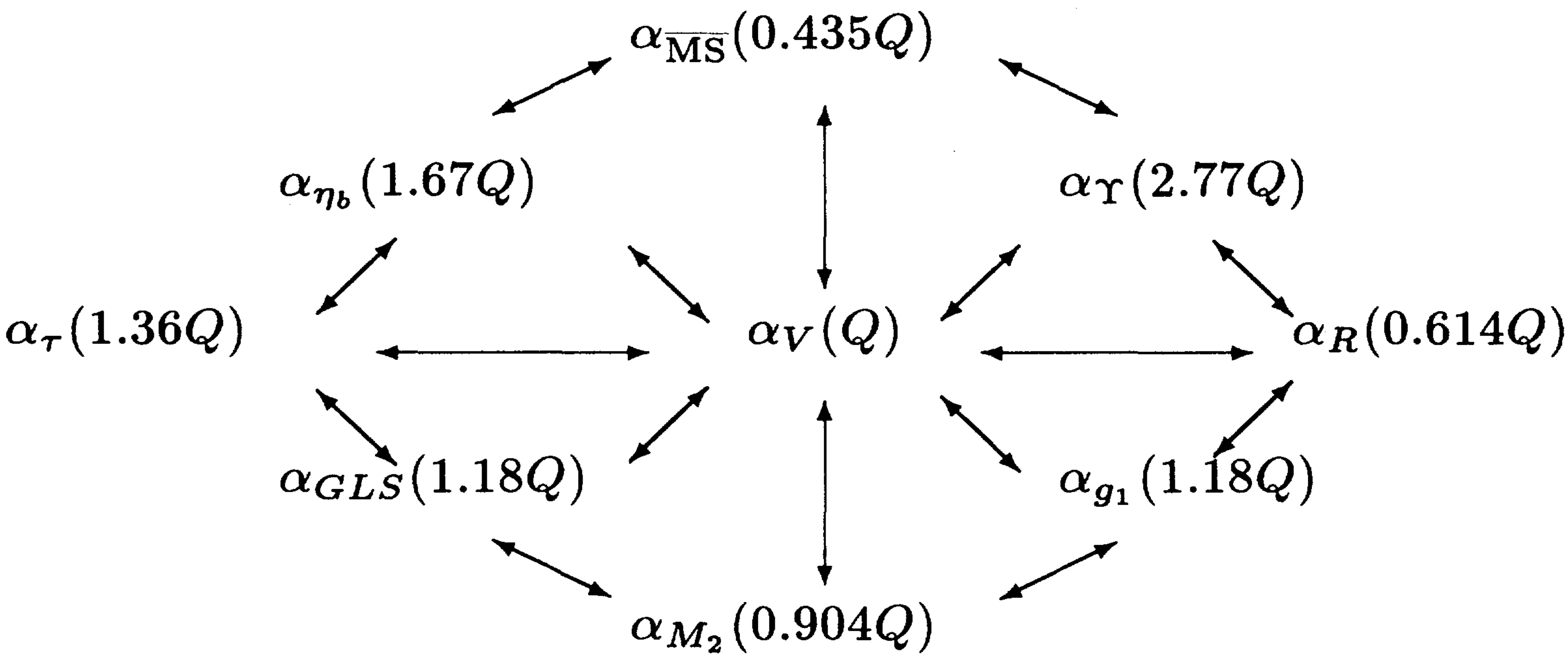
where $Q^* = e^{-5/6}$, $Q \cong 0.43Q$. This result shows that the effective scale of the $\overline{\rm MS}$ scheme should generally be about half of the true momentum transfer occurring in the interaction. In parallel to QED, the effective potential $V(Q^2)$ gives a particularly intuitive scheme for defining the QCD coupling constant

$$V(Q^2) \equiv -\frac{4\pi C_F \alpha_v(Q)}{Q^2} \tag{20}$$

Relate Observables to Each Other

- Eliminate Intermediate MSbar scheme
- No scale ambiguity
- Transitive!
- Commensurate Scale Relations
- Example: Generalized Crewther Relation

TABLE I. Leading order commensurate scale relations.



$$\frac{\alpha_R(Q)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(Q)}{\pi} + \left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^2 \left[\left(\frac{41}{8} - \frac{11}{3}\zeta_3\right) C_A - \frac{1}{8}C_F + \left(-\frac{11}{12} + \frac{2}{3}\zeta_3\right) f\right]$$
$$+ \left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^3 \left\{\left(\frac{90445}{2592} - \frac{2737}{108}\zeta_3 - \frac{55}{18}\zeta_5 - \frac{121}{432}\pi^2\right) C_A^2 + \left(-\frac{127}{48} - \frac{143}{12}\zeta_3 + \frac{55}{3}\zeta_5\right) C_A C_F - \frac{23}{32}C_F^2\right.$$
$$+ \left[\left(-\frac{970}{81} + \frac{224}{27}\zeta_3 + \frac{5}{9}\zeta_5 + \frac{11}{108}\pi^2\right) C_A + \left(-\frac{29}{96} + \frac{19}{6}\zeta_3 - \frac{10}{3}\zeta_5\right) C_F\right] f$$
$$\left. + \left(\frac{151}{162} - \frac{19}{27}\zeta_3 - \frac{1}{108}\pi^2\right) f^2 + \left(\frac{11}{144} - \frac{1}{6}\zeta_3\right) \frac{d^{abc}d^{abc}}{C_F d(R)} \frac{\left(\sum_f Q_f\right)^2}{\sum_f Q_f^2}\right\}. \quad (3.1$$

$$\frac{\alpha_{g_1}(Q)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(Q)}{\pi} + \left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^2 \left[\frac{23}{12}C_A - \frac{7}{8}C_F - \frac{1}{3}f\right]$$
$$+ \left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^3 \left\{\left(\frac{5437}{648} - \frac{55}{18}\zeta_5\right) C_A^2 + \left(-\frac{1241}{432} + \frac{11}{9}\zeta_3\right) C_A C_F + \frac{1}{32}C_F^2\right.$$
$$\left. + \left[\left(-\frac{3535}{1296} - \frac{1}{2}\zeta_3 + \frac{5}{9}\zeta_5\right) C_A + \left(\frac{133}{864} + \frac{5}{18}\zeta_3\right) C_F\right] f + \frac{115}{648}f^2\right\}.$$

Apply BLM -- Amazing Simplification

$$\int_0^1 dx \left[g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)\right] \equiv \frac{1}{3}\left|\frac{g_A}{g_V}\right|\left[1 - \frac{\alpha_{g_1}(Q)}{\pi}\right]$$

$$\frac{\alpha_{g_1}(Q)}{\pi} = \frac{\alpha_R(Q^*)}{\pi} - \left(\frac{\alpha_R(Q^{**})}{\pi}\right)^2 + \left(\frac{\alpha_R(Q^{***})}{\pi}\right)^3$$

Geometric Series in Conformal QCD

Generalized Crewther Relation

Lu, Kataev, Gabadadze, Sjb

H. Lu
A. Kataev
Gabadadze
SJB

Commensurate Scale Relations

relate observable to observable

no scale-ambiguity

$$R_{e^+e^-}(s) \equiv 3\sum e_q^2 \left[1 + \frac{\alpha_R(s)}{\pi}\right]$$

$$\int_0^1 dx \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right] \equiv \frac{g_A}{6}\left[1 - \frac{\alpha_{g_1}(Q^2)}{\pi}\right]$$

Find:

$$\left[1 + \frac{\alpha_R(s^*)}{\pi}\right]\left[1 - \frac{\alpha_{g_1}(Q^2)}{\pi}\right] = 1$$

all orders in perturbation theory! Abelian + Non-Abelian!

coefficients = "conformal" series

Crewther Relation!

$\beta \neq 0$ summed into scales

$$\sqrt{s^*} = 0.52\, Q + \cdots$$

* why is the relation between

α_R and α_{g_1}

so simple?

Gabadadze
Kataev
H.J. Lu
SJB

Consider conformal limit $\beta_0 \Rightarrow 0,\ \beta_1 \Rightarrow 0 \ \cdots$

$$\text{CSR} \Rightarrow \quad (1+\hat{\alpha}_R)(1-\hat{\alpha}_{g_1}) = 1$$

* Follows from Crewther relation!

$\beta=0$
chiral

$$3S = K R' = K\left(\tfrac{4}{3} R\right)$$

$\pi^0 \to \gamma\gamma$ — Adler Bardeen

β_g — GLS

$\hat{R}_{e^+e^-}$

$\check{R}_{e^+e^-}$

Deviations from Crewther Relation

proportional to β

Kataev
Broadhurst

Commensurate Scale Relations:

$$* \quad \alpha_B(Q_B) = \alpha_A(Q_A)\left[1 + C^{(1)}_{A/B}\frac{\alpha_A}{\pi} + \cdots\right]$$

conformal coeff.

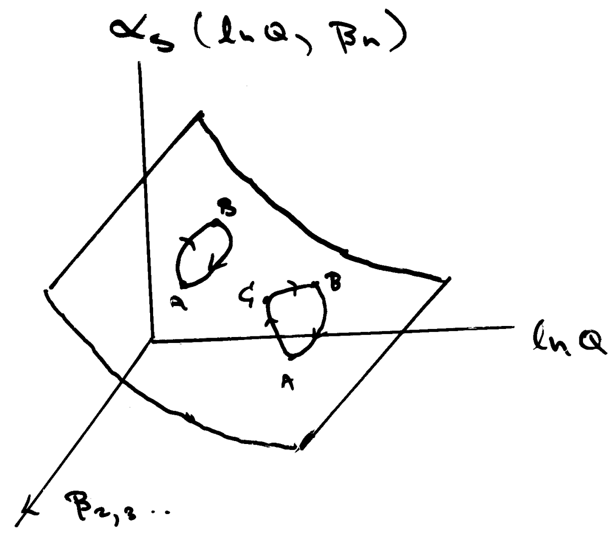
$$Q_B/Q_A = \lambda_{B/A}$$

Peterman
Stückelberg
Renormalization
"Group"

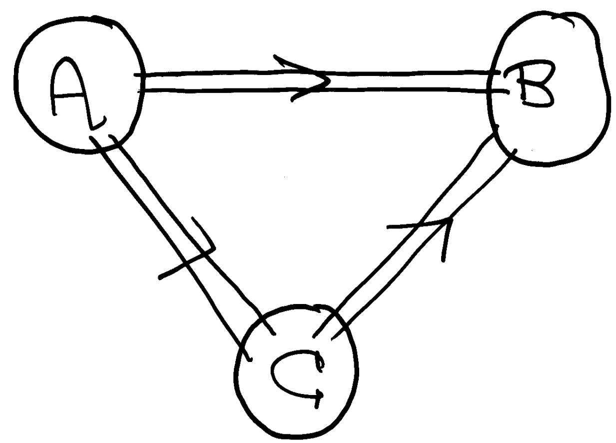
$$\lambda_{B/A} = \lambda_{B/C}/\lambda_{A/C} \qquad \text{transitive}$$

$$\lambda_{B/A} = \lambda^{-1}_{A/B} \qquad \text{symmetry}$$

$$\lambda_{A/A} = I \qquad \text{identity}$$



Transitivity property – Renormalization Group



$\boxed{A \Rightarrow C \Rightarrow B}$

same as $\boxed{A \Rightarrow B}$

indep of $\boxed{C}$

Relation between observables $A \Leftrightarrow B$

independent of choice of C

independent of scheme or

theoretical convention!

PMS violates transitivity

Transitivity of the renormalization group implies predictions for a physical observable $\mathcal{O}$ cannot depend on choice of intermediate renormalization scheme,

e.g., choice of $\alpha_{\overline{MS}}$ or α_{mom}.

$$\frac{d\mathcal{O}}{d\mu_{\text{scheme}}} = 0$$

not

$$\frac{d\mathcal{O}}{d\mu_{\text{renormalization}}} = 0$$

Kramer & Lampe:

PMS and FAC methods give unphysical scale dependence

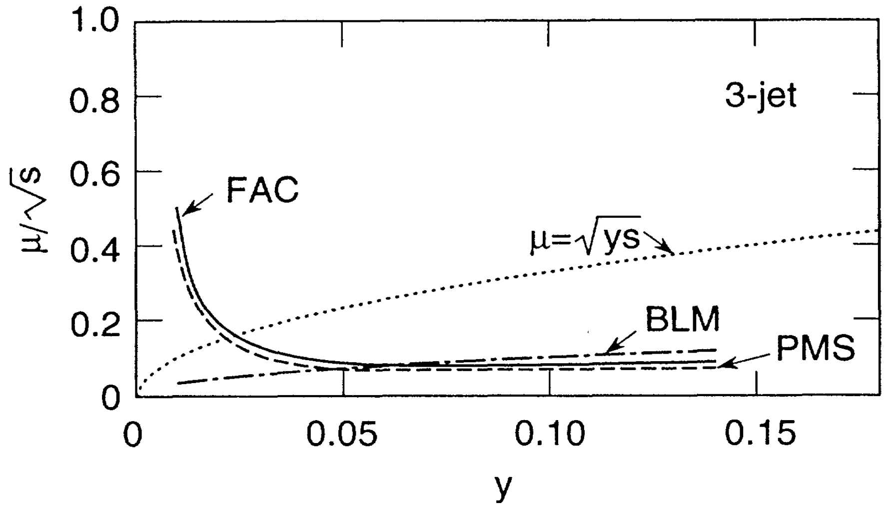


FIG. 1. The scale $\mu/\sqrt{s}$ according to the BLM (dashed-dotted), PMS (dashed), FAC (full), and $\sqrt{y}$ (dotted) procedures for the three-jet rate in e^+e^- annihilation, as computed by Kramer and Lampe [10]. Notice the strikingly different behavior of the BLM scale from the PMS and FAC scales at low y. In particular, the latter two methods predict increasing values of μ as the jet invariant mass $\mathcal{M} < \sqrt{(ys)}$ decreases.

These scale-setting methods can give strikingly different results in practical applications. For example, Kramer and Lampe have analyzed [10] the application of the FAC, PMS, and BLM methods for the prediction of jet production fractions in e^+e^- annihilation in PQCD. Jets are defined by clustering particles with invariant mass less than $\sqrt{ys}$, where y is the resolution parameter and $\sqrt{s}$ is the total center-of-mass energy. Physically, one expects the renormalization scale μ to reflect the invariant mass of the jets, that is, μ should be of order $\sqrt{ys}$. For example, in the analogous problem in QED, the maximum virtuality of the photon jet which sets the argument of the running coupling $\alpha(Q)$ cannot be larger than $\sqrt{ys}$. Thus one expects μ to decrease as the resolution parameter $y \to 0$. However, the scales chosen by the FAC and PMS methods do not reproduce this behavior (see Fig. 1): The predicted scales $\mu_{\rm PMS}(y)$ and $\mu_{\rm FAC}(y)$ rise without bound at small values for the jet fraction y. On the other hand, the BLM scale has the correct physical behavior as $y \to 0$. Since the argument of the running coupling becomes small using the BLM method, standard QCD perturbation theory in $\alpha_s[\mu_{\rm BLM}(y)]$ will not be convergent in the low y domain [11]. In contrast, the scales chosen by PMS and FAC give no sign that the perturbative results break down in the soft region.

Conventional renormalization scale-setting method:

- Guess arbitrary renormalization scale and take arbitrary range. Wrong for QED and Precision Electroweak.
- Prediction depends on choice of renormalization scheme
- Variation of result with respect to renormalization scale only sensitive to nonconformal terms; no information on genuine (conformal) higher order terms
- Conventional procedure has no scientific basis.
- FAC and PMS give unphysical results.
- Renormalization scale not arbitrary! Sets # active flavors

Use BLM!

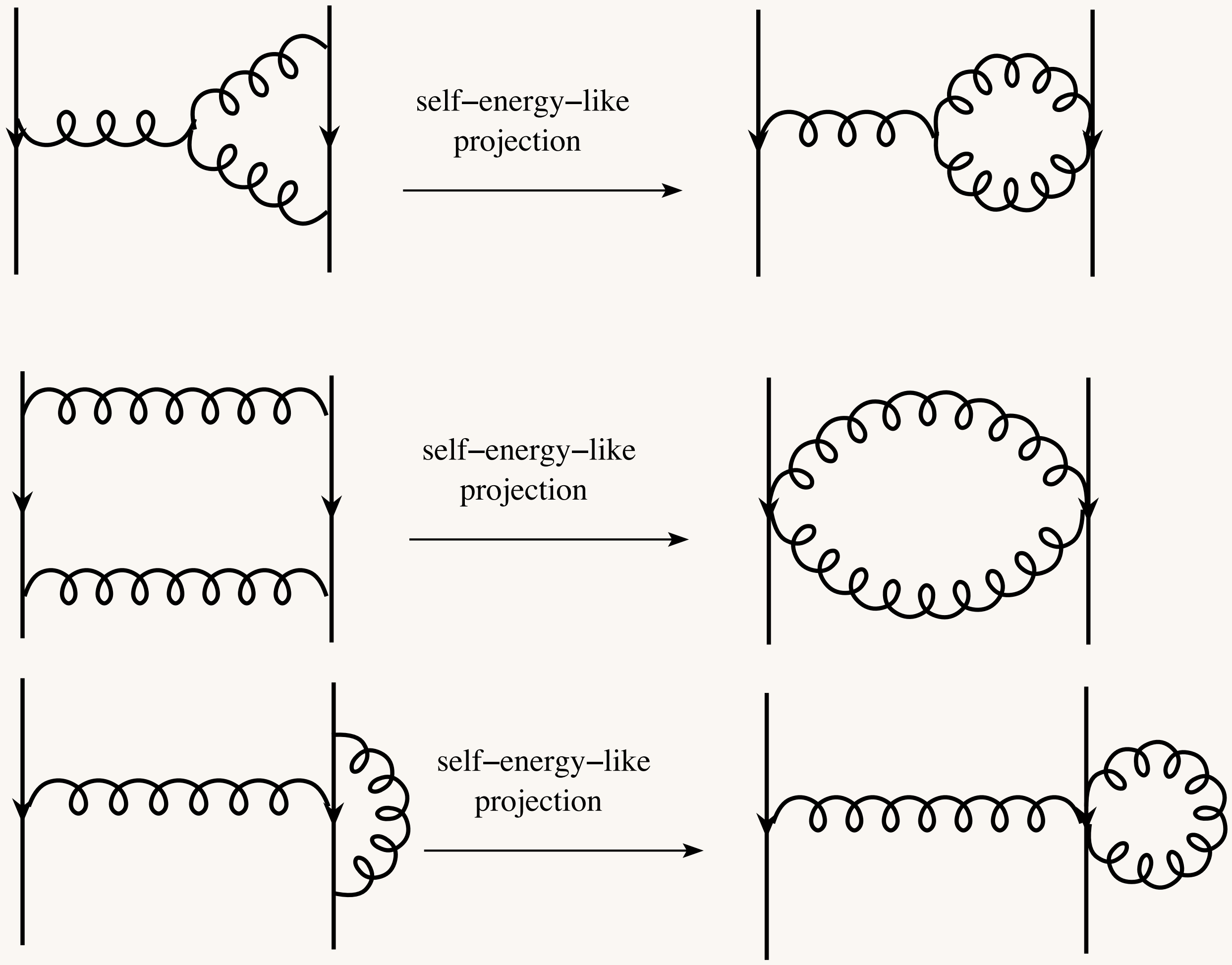
- Rigorous method: Satisfies Transitivity, all aspects of RG
- Preserves Conformal Template
- Physical Interpretation of Scales
- Correct Abelian Limit
- Eliminates unnecessary source of imprecision of PQCD predictions
- Commensurate Scale Relations: Fundamental Tests of QCD free of renormalization scale and scheme ambiguities
- BLM used in many applications, LGTH, BFKL, ...

Use Physical Scheme

- Use Observable to define QCD coupling or Pinch Scheme
- Analytic: Smooth behavior as one crosses new quark threshold
- New perspective on grand unification

Binger, Sjb

Pinch Scheme -- Effective Charge



Physical Renormalization Schemes and Grand Unification

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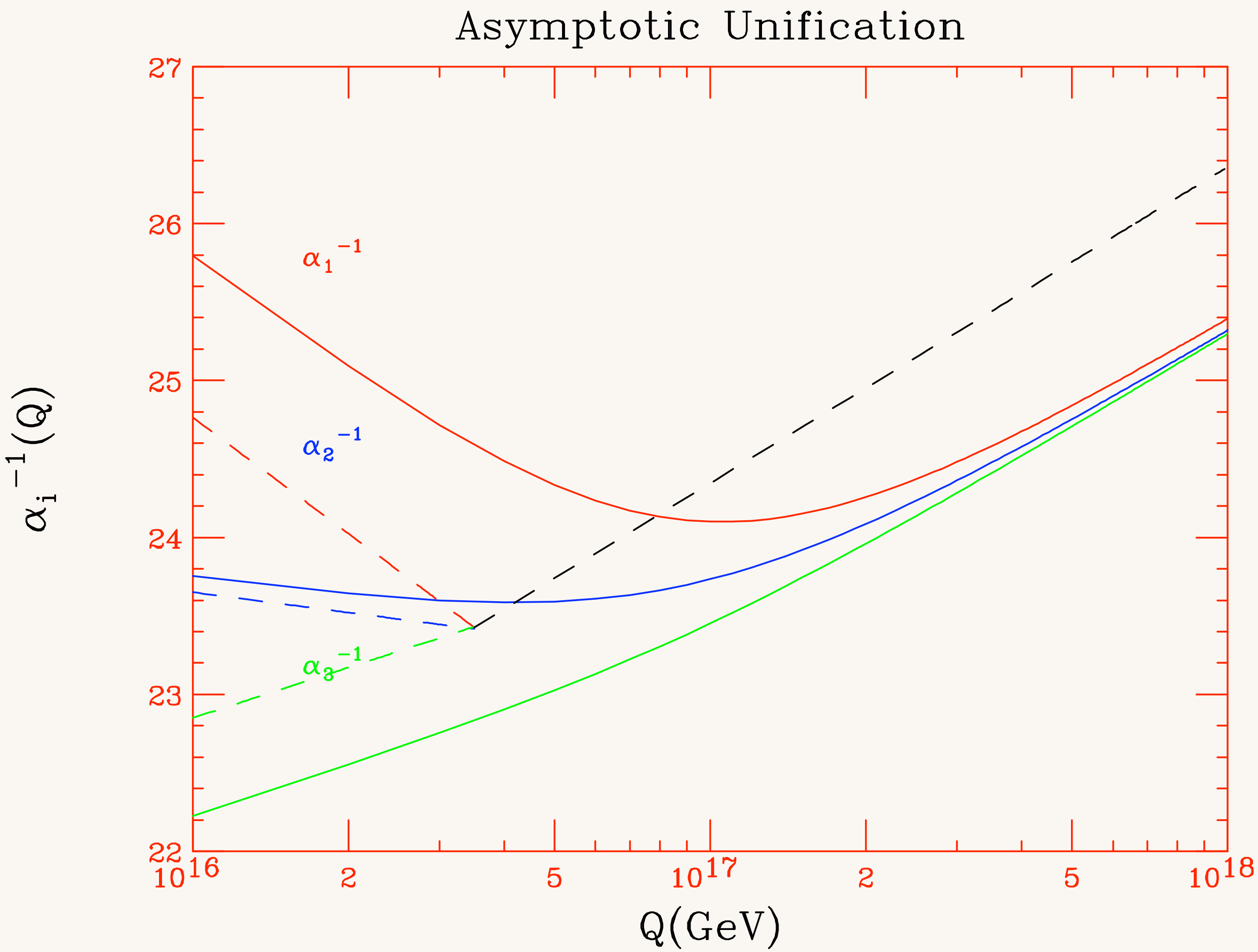
Abstract

In a physical renormalization scheme, gauge couplings are defined directly in terms of physical observables. Such effective charges are analytic functions of physical scales, and thus mass thresholds are treated with their correct analytic dependence. In particular, particles will contribute to physical predictions even at energies below their threshold. This is in contrast to unphysical renormalization schemes such as $\overline{MS}$ where mass thresholds are treated as step functions. In this paper we analyze supersymmetric grand unification in the context of physical renormalization schemes and find a number of qualitative differences and improvements in precision over conventional approaches. The effective charge formalism presented here provides a template for calculating all mass threshold effects for any given grand unified theory. These new threshold corrections may be important in making the measured values of the gauge couplings consistent with unification.



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Asymptotic Unification. The solid lines are the analytic $\overline{PT}$ effective couplings, while the dashed lines are the $\overline{DR}$ couplings. For illustrative purposes, $\alpha_3(M_Z)$ has been chosen so that unification occurs at a finite scale for $\overline{DR}$ and asymptotically for the $\overline{PT}$ couplings. Here $M_{SUSY} = 200\text{GeV}$ is the mass of all light superpartners except the wino and gluino which have values $\frac{1}{2}m_{\widetilde{g}} = M_{SUSY} = 2m_{\widetilde{w}}$. For illustrative purposes, we use $SU(5)$.

Factorization scale

$$\mu_{\text{factorization}} \neq \mu_{\text{renormalization}}$$

- Arbitrary separation of soft and hard physics
- Dependence on factorization scale not associated with beta function - present even in conformal theory
- Keep factorization scale separate from renormalization scale

$$\frac{d\mathcal{O}}{d\mu_{\text{factorization}}} = 0$$

- Residual dependence when one works in fixed order in perturbation theory.