Geometric Interpretation of Generalized Parton Distributions

or: what DVCS has to do with the distribution of partons in the transverse plane

PRD 62, 71503 (2000).

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- DIS $\longrightarrow \Im$ (forward Compton amplitude)
  $\overset{Bj}{\hookrightarrow}$ parton distributions (PDs)
- Deeply Virtual Compton Scattering (DVCS)
  $\overset{Bj}{\hookrightarrow}$ generalized parton distributions (GPDs)
- Physical interpretation of GPDs for $\xi = 0$ and $t \neq 0$ as Fourier transforms of impact parameter dependent PDs.

Motivation:

X.Ji, PRL **78**, 610 (1997):

$$\boxed{\text{DVCS}} \quad \Leftrightarrow \quad \boxed{\text{GPDs}} \quad \Leftrightarrow \quad \boxed{\vec{J}_q}$$

$\hookrightarrow$ GPDs are interesting physical observable!

But do GPDs have a simple physical interpretation?

Deep-inelastic scattering (DIS)

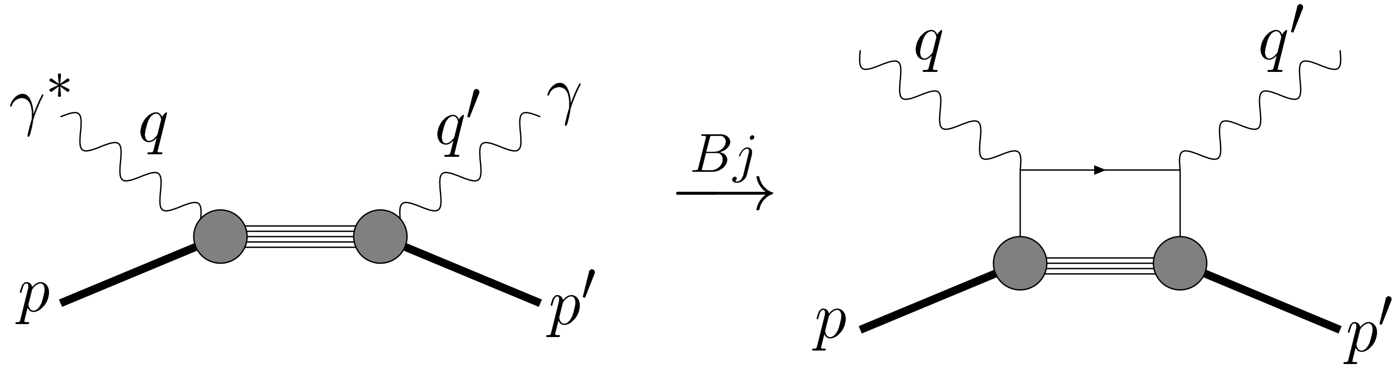
DIS $\longrightarrow$ (im. part of) forward Compton amplitude

$\overset{Bj}{\hookrightarrow}$ parton distributions

$$q(x) = \int \frac{dx^-}{4\pi} \left\langle P \left| \bar{q}(-\frac{x^-}{2}, \mathbf{0}_\perp)\gamma^+ q(\frac{x^-}{2}, \mathbf{0}_\perp) \right| P \right\rangle e^{ixp^+x^-}$$

No information on $\perp$ position of partons!

Deeply virtual Compton Scattering



probe for generalized parton distributions:

$$F_q(x,\xi,t) = \int \frac{dx^-}{2\pi} \langle P'|\overline{q}\left(-\frac{x^-}{2}\right)\gamma^+ q\left(\frac{x^-}{2}\right)|P\rangle\, e^{ix^- P^+ x}$$

with:

$$\bar{P}^\mu \equiv \frac{P^\mu + P'^\mu}{2}, \quad \Delta^\mu = P'^\mu - P^\mu, \quad \xi = \frac{\Delta^+}{\bar{P}^+}, \quad t = \Delta^2$$

$$T^{\mu\nu} = i\!\int d^4z e^{i\bar{q}\cdot z} \left\langle p' \left| T J^\mu\left(-\frac{z}{2}\right) J^\nu\left(\frac{z}{2}\right) \right| p \right\rangle$$

$$\overset{Bj}{\hookrightarrow} \frac{g_\perp^{\mu\nu}}{2} \int_{-1}^{1} dx \left(\frac{1}{x-\xi+i\varepsilon} + \frac{1}{x+\xi-i\varepsilon} \right) H(x,\xi,t,Q^2)\bar{u}(p')\gamma^+ u(p)$$

$$+ \ldots$$

$$\bar{q} = (q+q')/2 \qquad \bar{p} = (p+p')/2$$

$$x_{Bj} = -q^2/2p\cdot q \qquad x_{Bj} = 2\xi(1+\xi)$$

scaling functions:

$$\int \frac{dx^-}{2\pi} e^{ix^-\bar{p}^+x} \left\langle p' \left| \bar{q}\left(-\frac{x^-}{2}\right)\gamma^+ q\left(\frac{x^-}{2}\right) \right| p \right\rangle = H(x,\xi,\Delta^2)\bar{u}(p')\gamma^+ u(p) + E(x,\xi,\Delta^2)\bar{u}(p')\frac{i\sigma^{+\nu}\Delta_\nu}{2M}u(p)$$

for unpolarized DVCS, and

$$\int \frac{dx^-}{2\pi} e^{ix^-\bar{p}^+x} \left\langle p' \left| \bar{q}\left(-\frac{x^-}{2}\right)\gamma^+\gamma_5 q\left(\frac{x^-}{2}\right) \right| p \right\rangle = \tilde{H}(x,\xi,\Delta^2)\bar{u}(p')\gamma^+\gamma_5 u(p) + \tilde{E}(x,\xi,\Delta^2)\bar{u}(p')\frac{\gamma_5\Delta^+}{2M}u(p)$$

for polarized DVCS

parton interpretation:

"amplitude that parton with long. momentum $(x - \xi/2)\bar{p}^+$ is taken out of a nucleon with long momentum $(1 - \xi/2)\bar{p}^+$ and inserted back into the nucleon with long. momentum transfer $\Delta^+ = \xi\bar{p}^+$ and $\perp$ momentum transfer $\vec{\Delta}_\perp$"

compare: conventional PDFs, where parton is inserted back into nucleon <u>without</u> momentum transfer!

- resemble both form factors and parton distributions:
- involve same operator that is used to calculate conventional PDFs, except $p' \neq p$
- $\int_{-1}^{1} dx H(x,\xi,t) = F_1(t)$
- $\xi = 0$, $t = 0$ (no momentum transfer)
  $\hookrightarrow \quad H_q(x,0,0) = q(x)$
- GPDs allow to determine how much quarks with momentum fraction x contribute to form factor.
- Definition of GPDs resembles that of form factors

$$\left\langle p' \left| \hat{O} \right| p \right\rangle = H(x,\xi,\Delta^2)\bar{u}(p')\gamma^+ u(p) + E(x,\xi,\Delta^2)\bar{u}(p')\frac{i\sigma^{+\nu}\Delta_\nu}{2M}u(p)$$

with $\hat{O} \equiv \int \frac{dx^-}{2\pi} e^{ix^-\bar{p}^+ x}\bar{q}\left(-\frac{x^-}{2}\right)\gamma^+ q\left(\frac{x^-}{2}\right)$

- relation between PDs and GPDs similar to relation between a 'charge' and a 'form factor'
- If form factors can be interpreted as Fourier transforms of charge distributions in position space, what is the analogous physical interpretation for GPDs ?

In general, GPDs probe

$$F(x,\xi,t) = \frac{1}{2\bar{p}^+}\left[H(x,\xi,t)\bar{u}(p')\gamma^+ u(p) \right.$$
$$\left. + E(x,\xi,t)\bar{u}(p') i\sigma^{+\nu} q_\nu u(p)\right]$$

with $\xi = \frac{q^+}{\bar{p}^+}$.

This talk, focus on unpolarized target & $\xi = 0$, where

$$F(x,0,t) = H(x,0,t) \equiv H(x,t)$$

today's talk:

interpretation of GPDs for $\xi = 0$ but $\vec{\Delta}_\perp \neq 0$:

will show below that $H(x,\xi=0,t)$ and $\tilde{H}(x,\xi=0,t)$ have simple physical interpretation as

Fourier transfrom of impact parameter dependent PDs w.r.t. the impact parameter, i.e.

$$\boxed{\begin{array}{l} H(x,0,-\mathbf{\Delta}_\perp^2) = \int d^2\mathbf{b}_\perp q(x,\mathbf{b}_\perp) e^{-i\mathbf{\Delta}_\perp\cdot\mathbf{b}_\perp} \\ \tilde{H}(x,0,-\mathbf{\Delta}_\perp^2) = \int d^2\mathbf{b}_\perp \Delta q(x,\mathbf{b}_\perp) e^{-i\mathbf{\Delta}_\perp\cdot\mathbf{b}_\perp} \end{array}}$$

$\hookrightarrow$ measuring $H(x,\xi=0,t)$ and $\tilde{H}(x,\xi=0,t)$ allows determining $q(x,\vec{b}_\perp)$ and $\Delta q(x,\vec{b}_\perp)$!

Impact Parameter Dependent PDF:

- define state that is localized in $\perp$ position

$$|\psi_{loc}\rangle \equiv \left|p^+, \mathbf{R}_\perp = \mathbf{0}_\perp\right\rangle \equiv \mathcal{N} \int d^2\mathbf{p}_\perp \left|p^+, \mathbf{p}_\perp\right\rangle$$

  (using light-cone wave functions, one can show that this state has $\mathbf{R}_\perp \equiv \Sigma_i\, x_i \mathbf{b}_{\perp,i} = \mathbf{0}_\perp$)

- For such localized state define **impact parameter dependent PDF**

$$q(x, \mathbf{b}_\perp) \equiv \int \frac{dx^-}{4\pi} \langle\psi_{loc}|\, \bar{\psi}(-\frac{x^-}{2}, \mathbf{b}_\perp)\gamma^+\psi(\frac{x^-}{2}, \mathbf{b}_\perp)\, |\psi_{loc}\rangle\, e^{ixp^+x^-}$$

  (compare: working in CM frame in nonrel. physics)

- use transl. invariance to relate to same matrix element that appears in def. of GPDs

$$\langle\psi_{loc}|\, \bar{\psi}(-\frac{x^-}{2}, \mathbf{b}_\perp)\gamma^+\psi(\frac{x^-}{2}, \mathbf{b}_\perp)\, |\psi_{loc}\rangle$$

$$= |\mathcal{N}|^2 \int d^2\mathbf{p}_\perp \int d^2\mathbf{p}'_\perp \left\langle p^+, \mathbf{p}'_\perp\right|\bar{\psi}(-\frac{x^-}{2}, \mathbf{b}_\perp)\gamma^+\psi(\frac{x^-}{2}, \mathbf{b}_\perp)\left|p^+, \mathbf{p}_\perp\right\rangle$$

$$= |\mathcal{N}|^2 \int d^2\mathbf{p}_\perp \int d^2\mathbf{p}'_\perp \left\langle p^+, \mathbf{p}'_\perp\right|\bar{\psi}(-\frac{x^-}{2}, \mathbf{0}_\perp)\gamma^+\psi(\frac{x^-}{2}, \mathbf{0}_\perp)\left|p^+, \mathbf{p}_\perp\right\rangle \times e^{i\mathbf{b}_\perp\cdot(\mathbf{p}_\perp - \mathbf{p}'_\perp)},$$

$\hookrightarrow$ $q(x, \mathbf{b}_\perp)$ is Fourier transform of $H(x, 0, -\mathbf{\Delta}_\perp)$.

$$\boxed{q(x, \mathbf{b}_\perp) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} H(x, -\mathbf{\Delta}_\perp^2) e^{i\mathbf{b}_\perp\cdot\mathbf{\Delta}_\perp}}$$

- one can show that $q(x, \mathbf{b}_\perp)$ has physical interpretation of a density, i.e.

$$\boxed{\begin{array}{l} q(x, \mathbf{b}_\perp) \geq 0 \text{ for } x > 0 \\ q(x, \mathbf{b}_\perp) \leq 0 \text{ for } x < 0 \end{array}}$$

interpretation of $q(x,\mathbf{b}_\perp)$ as density:

- quark bilinear in twist-2 GPD can be expressed in terms of light-cone 'good' component $\psi_{(+)} \equiv \frac{1}{2}\gamma^-\gamma^+\psi$

$$\bar{\psi}'\gamma^+\psi = \sqrt{2}\bar{\psi}'_{(+)}\gamma^+\psi_{(+)}.$$

- expand $\psi_{(+)}$ in terms of canonical raising and lowering operators

$$\psi_{(+)}(x^-,\mathbf{x}_\perp) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int \frac{d^2\mathbf{k}_\perp}{2\pi} \sum_s \times \left[u_{(+)}(k,s)b_s(k^+,\mathbf{k}_\perp)e^{-ikx} + v_{(+)}(k,s)d_s^\dagger(k^+,\mathbf{k}_\perp)e^{ikx}\right],$$

with usual (canonical) equal light-cone time x^+ anticommutation relations, e.g.

$$\left\{b_r(k^+,\mathbf{k}_\perp), b_s^\dagger(q^+,\mathbf{q}_\perp)\right\} = \delta(k^+ - q^+)\delta(\mathbf{k}_\perp - \mathbf{q}_\perp)\delta_{rs}$$

and the normalization of the spinors is such that

$$\bar{u}_{(+)}(p,r)\gamma^+u_{(+)}(p,s) = 2p^+\delta_{rs}.$$

- Using for example $\bar{u}_{(+)}(p',r)\gamma^+u_{(+)}(p,s) = 2\sqrt{p^+p^{+\prime}}\delta_{rs}$, one finds for $x>0$

$$q(x,\mathbf{b}_\perp) = \mathcal{N}' \sum_s \int \frac{d^2\mathbf{k}_\perp}{2\pi} \int \frac{d^2\mathbf{k}'_\perp}{2\pi} \langle\psi_{loc}|\, b_s^\dagger(xp^+,\mathbf{k}'_\perp)b_s(xp^+,\mathbf{k}_\perp)\,|\psi_{loc}\rangle \times e^{i\mathbf{b}_\perp\cdot(\mathbf{k}'_\perp - \mathbf{k}_\perp)}.$$

- Fourier transform to $\perp$ position space

$$\tilde{b}_s(k^+,\mathbf{x}_\perp) \equiv \int \frac{d^2\mathbf{k}_\perp}{2\pi} b_s(k^+,\mathbf{k}_\perp)e^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}$$

$\hookrightarrow$

$$\begin{aligned} q(x, \mathbf{b}_\perp) &= \sum_s \langle \psi_{loc} | \, \tilde{b}_s^\dagger(xp^+, \mathbf{b}_\perp) \tilde{b}_s(xp^+, \mathbf{b}_\perp) \, | \psi_{loc} \rangle \, . \\ &= \sum_s \left| \tilde{b}_s(xp^+, \mathbf{b}_\perp) \, | \psi_{loc} \rangle \right|^2 \\ &\geq 0. \end{aligned}$$

Overlap Representation for GPDS at $\xi = 0$: [1]

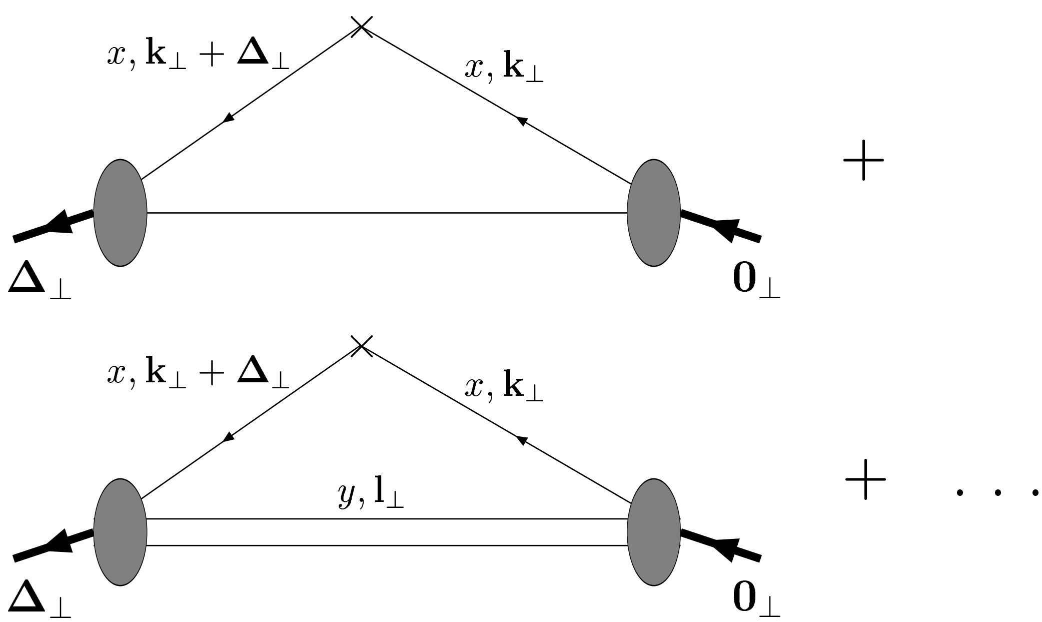
express GPDs for $\xi = 0$ as overlap integrals between LF wave functions (Fock space amplitudes) $\Psi_N(x, \mathbf{k}_\perp)$ (exact if one knows the Ψ_N for *all* Fock components)

$$H(x, 0, -\mathbf{\Delta}_\perp^2) = \sum_N \sum_j \int [dx]_N \int \left[d^2\mathbf{k}_\perp\right]_N \delta(x - x_j) \Psi_N^*(x_i, \mathbf{k}'_{\perp,i}) \Psi_N(x_i, \mathbf{k}_{\perp,i})$$

$[\ \mathbf{k}'_{\perp,i} = \mathbf{k}_{\perp,i} - x_i \mathbf{\Delta}_\perp$ for $i \neq j$ and $\mathbf{k}'_{\perp,j} = \mathbf{k}_{\perp,j} + (1 - x_j)\mathbf{\Delta}_\perp]$

Example:

$H(x, \vec{\Delta}_\perp)$ for the π



$$\begin{aligned} H(x, \mathbf{\Delta}_\perp) &= \int d^2\mathbf{k}_\perp \psi^*_{\mathbf{\Delta}_\perp}(x, \mathbf{k}_\perp + \mathbf{\Delta}_\perp) \psi_{\mathbf{0}_\perp}(x, \mathbf{k}_\perp) \\ &+ \int d^2\mathbf{k}_\perp d^2\mathbf{l}_\perp dy \psi^*_{\mathbf{\Delta}_\perp}(x, \mathbf{k}_\perp + \mathbf{\Delta}_\perp, y, \mathbf{k}_\perp) \psi_{\mathbf{0}_\perp}(x, \mathbf{k}_\perp, y, \mathbf{l}_\perp) \\ &+ \quad . \quad . \quad . \end{aligned}$$

[1] M.Diehl et al., NPB 596, 33 (2001); Same as form factor in Drell-Yan frame in terms of LF wave functions, except that x of 'active' quark is not integrated over.

compare nonrelativistic (NR) form factor

- 2-body system:

$$F(\vec{q}) = \int d^3\vec{k}\psi^*_{\vec{q}}(\vec{k}+\vec{q})\psi_{\vec{0}}(\vec{k})$$

- 3-body system:

$$F(\vec{q}) = \int d^3\vec{k}_1 d^3\vec{k}_2\, \psi^*_{\vec{q}}(\vec{k}_1+\vec{q},\vec{k}_2)\, \psi_{\vec{0}}(\vec{k}_1,\vec{k}_2)$$

Note: like GPDs, $F(\vec{q})$ also off-diagonal in momentum ($\vec{P}_{in} \neq \vec{P}_{out}$)

use: NR boosts purely kinematic

$$\vec{k}_i' = \vec{k}_i + x_i\vec{q}$$

with $x_i = \frac{m_i}{M}$

$\hookrightarrow$ simple boost properties of NR wave functions, e.g.

$$\begin{aligned} \psi_{\vec{q}}(\vec{k}) &= \psi_{\vec{0}}(\vec{k}-x\vec{q}) \\ \psi_{\vec{q}}(\vec{k}_1,\vec{k}_2) &= \psi_{\vec{0}}(\vec{k}_1-x_1\vec{q},\vec{k}_2-x_2\vec{q}) \end{aligned}$$

to rewrite $F(\vec{q})$ as autocorrelation of wf for$\vec{P}=\vec{0}$:

2-body system:

$$F(\vec{q}) = \int d^3\vec{k}\, \psi^*_{\vec{0}}(\vec{k}+(1-x)\vec{q})\, \psi_{\vec{0}}(\vec{k})$$

3-body system:

$$F(\vec{q}) = \int d^3\vec{k}_1 d^3\vec{k}_2\, \psi^*_{\vec{0}}(\vec{k}_1+(1-x_1)\vec{q},\vec{k}_2-x_2\vec{q})\, \psi_{\vec{0}}(\vec{k}_1,\vec{k}_2)$$

2. Fourier transform to position space to transform autocorrelation function into density, i.e. to express $F(\vec{q})$ in terms of the charge density

$$F(\vec{q}) = \int d^3 r e^{-i\vec{q}\cdot\vec{r}} \rho(\vec{r}),$$

where

$\rho(\vec{r}) =$ charge distribution measured from the **center of mass** $\vec{R}_{CM} \equiv \Sigma_i \frac{m_i}{M} \vec{r}_i$

relevance for GPDs

- purely transverse boosts in LF frame form Galilean subgroup

$$\begin{aligned} x_i &\longrightarrow x_i \\ \mathbf{k}_{\perp,i} &\longrightarrow \mathbf{k}_{\perp,i} + x_i \boldsymbol{\Delta}_\perp \end{aligned}$$

  where momentum fraction x_i plays role of mass fraction $\frac{m_i}{M}$ in NR case

$\hookrightarrow$ LF Fock space amplitudes transform under purely $\perp$ boosts very similar to the way NR wave functions transform

$$\begin{aligned} \psi_{\boldsymbol{\Delta}_\perp}(x, \mathbf{k}_\perp) &= \psi_{\mathbf{0}_\perp}(x, \mathbf{k}_\perp - x\boldsymbol{\Delta}_\perp) \\ \psi_{\boldsymbol{\Delta}_\perp}(x, \mathbf{k}_\perp, y, \mathbf{l}_\perp) &= \psi_{\mathbf{0}_\perp}(x, \mathbf{k}_\perp - x\boldsymbol{\Delta}_\perp, y, \mathbf{l}_\perp - y\boldsymbol{\Delta}_\perp) \end{aligned}$$

$\hookrightarrow$ can represent $H(x, \boldsymbol{\Delta}_\perp)$ as autocorrelation of Fock space amplitudes in $\mathbf{p}_\perp = \mathbf{0}_\perp$ frame

$\hookrightarrow$ Fourier transform (the $\perp$ coordinates) to position space, yielding (compare $F(\vec{q}) \Leftrightarrow \rho(\vec{r})$ in NR QM)

$$H(x, \boldsymbol{\Delta}_\perp) = \int d^2\mathbf{b}_\perp e^{-i\boldsymbol{\Delta}_\perp \cdot \mathbf{b}_\perp} q(x, \mathbf{b}_\perp)$$

where

$q(x, \mathbf{b}_\perp) =$ probability density to find quark with momentum fraction x at $\perp$ distance $\mathbf{b}_\perp$ from the $\perp$ center of momentum $\mathbf{R}_\perp^{CM} \equiv \Sigma_i x_i \mathbf{r}_{i,\perp}$.

- formal def. for $q(x, \mathbf{b}_\perp)$ (in LF gauge):

$$q(x, \mathbf{b}_\perp) \;=\; \int \frac{dx^-}{4\pi} e^{ip^+x^-x} \left\langle \psi_{loc} \left| \bar{q}\left(-\frac{x^-}{2}, \mathbf{b}_\perp\right) \gamma^+ q\left(\frac{x^-}{2}, \mathbf{b}_\perp\right) \right| \psi_{loc} \right\rangle$$

  where $|\psi_{loc}\rangle \equiv \int d^2p_\perp \psi(\mathbf{p}_\perp) |p\rangle$ is a wave packet of plane wave proton states which is very localized in the $\perp$ direction, but still has a sharp P^+ total $\perp$ momentum!) is at the origin.

- other gauges: inserts straight line gauge string from $(-\frac{x^-}{2}, \mathbf{b}_\perp)$ to$(\frac{x^-}{2}, \mathbf{b}_\perp)$
  $\hookrightarrow$ manifestly gauge invariant!

Discussion:

1. $H(x, -\mathbf{\Delta}_\perp^2)$ tells us (via Fourier trafo) how partons are distributed in the transverse plane as a function of the distance from the $\perp$ center of momentum.
2. similar interpretation exists for $\tilde{H}(x, -\mathbf{\Delta}_\perp^2)$.
3. $\mathbf{b}_\perp$ distribution is measured w.r.t. $\mathbf{R}_\perp^{CM} \equiv \Sigma_i\, x_i \mathbf{r}_{i,\perp}$
   $\hookrightarrow$ width of the $\mathbf{b}_\perp$ distribution should go to zero as $x \to 1$, since the active quark becomes the $\perp$ center of momentum in that limit!
   $\hookrightarrow$ $H(x,t)$ should become t-independent as $x \to 1$.
4. $q(x, \mathbf{b}_\perp)$ has probablilistic interpretation:

$$\begin{aligned} q(x,\mathbf{b}_\perp) &\sim \left\langle \psi_{loc} \left| b^\dagger(xp^+,\mathbf{b}_\perp) b(xp^+,\mathbf{b}_\perp) \right| \psi_{loc} \right\rangle \\ &= \left| b(xp^+,\mathbf{b}_\perp)|\psi_{loc}\rangle \right|^2 , \end{aligned}$$

   where $b(xp^+, \mathbf{b}_\perp)$ creates quarks of long. momentum xp^+ at $\perp$ position $\mathbf{b}_\perp$.
   $\hookrightarrow$ positivity constraint

$$0 < q(x,\mathbf{b}_\perp) \equiv \int d^2\mathbf{\Delta}_\perp H_q(x,0,-\mathbf{\Delta}_\perp^2) e^{i\mathbf{\Delta}_\perp\cdot\mathbf{b}_\perp}$$

   for $x > 0$ and negative for $x < 0$.
5. Use intuition about nucleon structure in position space to make predictions for GPDs:

large x: quarks expected to come from localized valence 'core',
small x also contributions larger ' meson cloud'
$\hookrightarrow$ expect a gradual increase of the t-dependence of $H(x,0,t)$ as one goes from larger to smaller values of x

6. commonly used ansatz for t dependence (motivated by LF constituent models with Gaussian wf):

$$H(x,0,-\mathbf{\Delta}_\perp^2) = q(x)e^{-a\mathbf{\Delta}_\perp^2 \frac{1-x}{x}}$$

$\langle {\mathbf{b}_\perp}^2\rangle \sim \frac{1}{x}$
inconsistent with space time descriptions of parton structure @ small x (Gribov)

$$\langle {\mathbf{b}_\perp}^2\rangle \sim \alpha \ln\frac{1}{x}$$

$\hookrightarrow$ LF const. model with Gaussian wf no good @ small x.

7. Better ansatz:

$$H(x,0,-\mathbf{\Delta}_\perp^2) = q(x)e^{-\alpha\mathbf{\Delta}_\perp^2 \ln\frac{1}{x}} \quad \text{or} \quad q(x)e^{-\alpha\mathbf{\Delta}_\perp^2 (1-x)\ln\frac{1}{x}}$$

(2^{nd} ansatz also consistent with Drell-Yan-West).

Form factors $\leftrightarrow$ (Fourier trafo of) charge distributions

- fixed target: $\mathcal{EM}$ form factor $\rightarrow$ Fourier trafo of charge distribution
- 'moving' target
  - separate center of mass motion (nonrelativistic!)
  - form localized wave packet
    $\hookrightarrow$ (uncertainty principle) relativistic corrections
    relevant scale: $\lambda_C \sim \frac{1}{M}$

- wave packet

$$|\Psi\rangle = \int \frac{d^3p \;\; \psi(\vec{p})}{\sqrt{2E_{\vec{p}}(2\pi)^3}} \, |\vec{p}\rangle \, ,$$

- $E_{\vec{p}} = \sqrt{M^2 + \vec{p}^2}$ and covariant normalization $\langle \vec{p}' | \vec{p} \rangle = 2E_{\vec{p}} \delta(\vec{p}' - \vec{p})$
- charge distribution in the wave packet

$$\begin{aligned} \mathcal{F}_\psi(\vec{q}) &\equiv \int d^3x e^{-i\vec{q}\cdot\vec{x}} \langle \Psi | \, \rho(\vec{x}) \, |\Psi\rangle \\ &= \int \frac{d^3p}{\sqrt{2E_{\vec{p}} 2E_{\vec{p}'}}} \Psi^*(\vec{p}+\vec{q})\Psi(\vec{p}) \langle \vec{p}' | \, \rho(\vec{0}) \, |\vec{p}\rangle \\ &= \frac{1}{2} \int d^3p \frac{E_{\vec{p}} + E_{\vec{p}'}}{\sqrt{E_{\vec{p}} E_{\vec{p}'}}} \Psi^*(\vec{p}+\vec{q})\Psi(\vec{p}) F(q^2). \end{aligned}$$

- Nonrelativistic case:

  –
  $$\frac{E_{\vec{p}} + E_{\vec{p}'}}{2\sqrt{E_{\vec{p}} E_{\vec{p}'}}} = 1$$

  –
  $$q^2 = -\vec{q}^2$$

  – charge distribution in the wave packet

  $$\mathcal{F}_\psi(\vec{q}) = \int d^3p \Psi^*(\vec{p}+\vec{q})\Psi(\vec{p}) F(\vec{q}^2)$$

  – choose $\Psi(\vec{p})$ very localized in position space $\Psi^*(\vec{p}+\vec{q}) \approx \Psi^*(\vec{p})$

  $$\hookrightarrow \qquad F_\Psi(\vec{q}) = F(\vec{q}^2)$$

- Relativistic corrections (example rms radius):

$$\mathcal{F}_\Psi(\vec{q}^2) = 1 - \frac{R^2}{6}\vec{q}^2 - \frac{R^2}{6}\int d^3p\,|\Psi(\vec{p})|^2 \frac{(\vec{q}\cdot\vec{p})^2}{E_{\vec{p}}^2} + \int d^3p \left|\vec{q}\cdot\vec{\nabla}\Psi(\vec{p})\right|^2 - \frac{1}{8}\int d^3p\,|\Psi(\vec{p})|^2 \frac{(\vec{q}\cdot\vec{p})^2}{E_{\vec{p}}^4},$$

(R^2 defined as usual: $F(q^2) = 1 + \frac{R^2}{6}q^2 + \mathcal{O}(q^4)$)

If one localizes the wave packet, i.e.

$$\int d^3p \left|\vec{q}\cdot\vec{\nabla}\Psi(\vec{p})\right|^2 \to 0,$$

then relativistic corrections diverge ($\Delta x \Delta p \sim 1$)

$$\frac{R^2}{6}\int d^3p\,|\Psi(\vec{p})|^2 \frac{(\vec{q}\cdot\vec{p})^2}{E_{\vec{p}}^2} \to \infty$$

$$\frac{1}{8}\int d^3p\,|\Psi(\vec{p})|^2 \frac{(\vec{q}\cdot\vec{p})^2}{E_{\vec{p}}^4} \to \infty$$

- in rest frame, wave packet + rel. corrections contribute at least $\Delta R^2 \sim \lambda_C^2 = \frac{Q^2}{M^2}$
  identification of charge distribution in rest frame with Fourier transformed form factor only unique down to scale λ_C

- infinite momentum frame: rel. corrections governed by $\frac{\vec{p}\cdot\vec{q}}{E_{\vec{p}}^2}$ and $\frac{\vec{q}^2}{E_{\vec{p}}^2}$

  consider wave packet $\Psi(\vec{p}_\perp)$ in transverse direction, with

– sharp longitudinal momentum $P_z \to \infty$

– transverse size of wave packet $r_\perp$, with $R \gg r_\perp \gg \frac{1}{P_z}$

take momentum transfer purely transverse

$\hookrightarrow \quad \mathcal{F}_\Psi(\vec{q}_\perp) = F(\vec{q}_\perp^{\,2})$

$\hookrightarrow$ form factor can be interpreted as Fourier transform of charge distribution w.r.t. impact parameter in ∞ momentum frame (without λ_C uncertainties!)

GPDs for $\xi = 0$

- consider wave packet in $\perp$ $(x-y)$ direction (P_z fixed)

$$|\Psi\rangle = \int \frac{d^2p \;\; \psi(\vec{p})}{\sqrt{2E_{\vec{p}}(2\pi)^2}} |\vec{p}\rangle \,,$$

- define (usual) parton distribution in this wave packet as function of impact parameter $\vec{b}_\perp$

$$f_\Psi(x,\vec{b}_\perp) \equiv \int \frac{dx^-}{2\pi} e^{ixp^+x^-} \left\langle \Psi \left| \bar{q}\left(-\frac{x^-}{2},\vec{b}_\perp\right)\gamma^+ q\left(\frac{x^-}{2},\vec{b}_\perp\right) \right| \Psi \right\rangle$$

- in the following: show that (for localized Ψ) Fourier trafo of $f_\Psi(x,\vec{b}_\perp)$ w.r.t. $\vec{b}_\perp$ yields $H(x,\xi=0,t)$.

$$\begin{aligned}
\mathcal{F}_\Psi(x,\vec{q}_\perp) &\equiv \int d^2q_\perp e^{-i\vec{q}_\perp\vec{x}_\perp} f_\Psi(x,\vec{x}_\perp) \\
&= \int \frac{d^2p_\perp \Psi^*(\vec{p}'_\perp)\Psi(\vec{p}_\perp)}{\sqrt{2E_{\vec{p}}2E_{\vec{p}'}}} \int \frac{dx^-}{4\pi} e^{ixp^+x^-} \langle p'|\bar{q}(-\frac{x^-}{2},\vec{0}_\perp)q(\frac{x^-}{2},\vec{0}_\perp)|p\rangle \\
&= \int \frac{d^2p_\perp \Psi^*(\vec{p}'_\perp)\Psi(\vec{p}_\perp)}{\sqrt{2E_{\vec{p}}2E_{\vec{p}'}}} f(x,\xi=0,q^2).
\end{aligned}$$

where $p'_z = p_z$ and $\vec{p}'_\perp = \vec{p}_\perp + \vec{q}_\perp$

- nonrelativistic case ($q^2 = -\vec{q}^2$ and $E_{\vec{p}} = E_{\vec{p}'} = m$)
  $\hookrightarrow \quad H(x,\xi=0,-\vec{q}_\perp^2)$ not dependent on $\vec{p}$
  $\hookrightarrow$ take out of integral
  Furthermore, if size of $\Psi(\vec{p}_\perp)$ in $\perp$ position space is

taken to zero then $\int \frac{d^2p_\perp \Psi^*(\vec{p}'_\perp)\Psi(\vec{p}_\perp)}{2E_{\vec{p}}2} \rightarrow \frac{1}{2M}$

$\hookrightarrow$

$$\mathcal{F}_\Psi(x,\vec{q}_\perp) \rightarrow \frac{1}{2M} H(x,\xi=0,-\vec{q}^2)$$

dependence on wave packet disappears if very localized!

$\hookrightarrow$ $H(x,\xi=0,-\vec{q}^2)$ has interpretation as Fourier transform of impact parameter dependent parton distribution w.r.t. impact parameter

- relativistic case (rest frame):
  same interpretation, but resolution in $\vec{b}_\perp$ of order λ_C

- relativistic case (∞ momentum frame):

$$\mathcal{F}_\Psi(x,\vec{q}_\perp) = \int \frac{d^2p_\perp \Psi^*(\vec{p}'_\perp)\Psi(\vec{p}_\perp)}{\sqrt{2E_{\vec{p}}2E_{\vec{p}'}}} H(x,\xi=0,q^2).$$

infinite momentum frame (again wave packet with $r_\perp$ such that ($\frac{1}{p_z} \ll r_\perp \ll R$)

$\hookrightarrow \qquad q^2 = -\vec{q}^2 \qquad$ and $\qquad E_{\vec{p}} = E_{\vec{p}'} = p_z$

$\hookrightarrow$

$$\mathcal{F}_\Psi(x,\vec{q}_\perp) \to \frac{1}{2p_z} H(x,\xi=0,-\vec{q}^2)$$

$\hookrightarrow$ in ∞ momentum frame no relativistic corrections to naive interpretation

Conclusion: Since Fourier trafo[2] of $\vec{b}_\perp$-dependent PDF for target that is localized in $\perp$ direction agrees with GPD for $\xi = 0$, we can identify

$$\boxed{\begin{aligned} H(x,0,-\vec{\Delta}_\perp^2) &= \int d^2b_\perp q(x,\vec{b}_\perp)e^{-i\vec{b}_\perp\vec{\Delta}_\perp} \\ \tilde{H}(x,0,-\vec{\Delta}_\perp^2) &= \int d^2b_\perp \Delta q(x,\vec{b}_\perp)e^{-i\vec{b}_\perp\vec{\Delta}_\perp} \end{aligned}}$$

$\hookrightarrow$ measuring $H(x,\xi=0,t)$ and $\tilde{H}(x,\xi=0,t)$ allows determining $q(x,\vec{b}_\perp)$ and $\Delta q(x,\vec{b}_\perp)$!

[2]Fourier trafo with respect to $\vec{b}_\perp$

QCD-evolution:

so far ignored QCD evolution! However, can be easily included

- For $t \ll Q^2$, leading order evolution t-independent
- For $\xi = 0$ evolution kernel for GPDs same as DGLAP evolution kernel

likewise:

- impact parameter dependent PDFs evolve such that different $\vec{b}_\perp$ do not mix (as long as $\perp$ spatial resolution much smaller than Q^2)

$\hookrightarrow$ above results consistent with QCD evolution:

$$\boxed{\begin{array}{l} H(x,0,-\vec{\Delta}_\perp^2,Q^2) = \int d^2b_\perp q(x,\vec{b}_\perp,Q^2) e^{-i\vec{b}_\perp\vec{\Delta}_\perp} \\ \tilde{H}(x,0,-\vec{\Delta}_\perp^2,Q^2) = \int d^2b_\perp \Delta q(x,\vec{b}_\perp,Q^2) e^{-i\vec{b}_\perp\vec{\Delta}_\perp} \end{array}}$$

where QCD evolution of $H, \tilde{H}, q, \Delta q$ is described by DGLAP and is independent on both $\vec{b}_\perp$ and $\vec{\Delta}_\perp^2$.

extrapolating to $\xi = 0$

- bad news: $\xi = 0$ <u>not</u> directly accessible in DVCS since long. momentum transfer necessary to convert virtual γ into real γ
- good news: moments of GPDs have simple ξ-dependence (polynomials in ξ) $\hookrightarrow$ should be possible to extrapolate!

even moments of $H(x, \xi, t)$:

$$H_n(\xi, t) \equiv \int_{-1}^{1} dx x^{n-1} H(x, \xi, t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} A_{n,2i}(t) \xi^{2i} + C_n(t)$$
$$= A_{n,0}(t) + A_{n,2}(t)\xi^2 + ... + A_{n,n-2}(t)\xi^{n-2} + C_n(t)\xi^n,$$

i.e. for example

$$\int_{-1}^{1} dx x H(x, \xi, t) = A_{2,0}(t) + C_2(t)\xi^2. \quad (1)$$

- For n^{th} moment, need $\frac{n}{2}+1$ measurements of $H_n(\xi, t)$ for same t but different ξ to determine $A_{n,2i}(t)$.
- $H_n(\xi = 0, t)$ obtained from

$$H_n(\xi = 0, t) = A_{n,0}(t)$$

- similar prodecure exists for moments of $\tilde{H}$

models:

most models for parton structure of hadrons also make predictions about distribution of partons in $\perp$ plane

- pion-cloud model for nucleon (qualitative):

  model: nonperturbative (low Q^2) sea quarks described by π-cloud surrounding a 'bare' nucleon.

  - Consequences for parton structure:
  - large x (valence quarks; from 'bare' nucleon) partons very localized in position space
  - small x (sea quarks; from π cloud) delocalized in position space

  $\hookrightarrow$ more localization in $\mathbf{b}_\perp$ for increasing x
  $\hookrightarrow$ less t dependence in $H(x,0,t)$ for increasing x

- phenomenological LF-wave functions with Gaussian ansatz for $\mathbf{k}_\perp$-dependence

$$\Psi_N(x_i, \mathbf{k}_{\perp,i}) \propto \exp\left[-a^2 \sum_{i=1}^{N} \frac{\mathbf{k}_{\perp,i}^2}{x_i}\right]$$

  Insert into overlap integral for $H(x,\xi=0,t)$ ... $\Rightarrow$

$$H(x,t) = q(x) \exp\left[\frac{a^2}{2}\frac{1-x}{x}t\right]$$

$\hookrightarrow$

$$q(x,\mathbf{b}_\perp) = q(x)\frac{x}{2\pi a^2(1-x)}\exp\left[-\frac{1}{2a^2}\frac{x}{1-x}\mathbf{b}_\perp^2\right]$$

very localized (in $\mathbf{b}_\perp$) for $x \to 1$!

too large ($\perp$ size $\sim \frac{1}{x}$) for $x \to 0$

- NJL-model (for $H(x,t)$ in the pion)

$$H(x,0,t) = 1 - \frac{3g^2\vec{\Delta}_\perp^2(1-x)^2}{8\pi^2}\int_0^1 d\alpha \left[\frac{1}{[M^2+\vec{\Delta}_\perp^2(1-x)^2\alpha(1-\alpha)]} - \frac{1}{[M^2+\Lambda^2+\vec{\Delta}_\perp^2(1-x)^2\alpha(1-\alpha)]} - \frac{\Lambda^2}{[M^2+\Lambda^2+\vec{\Delta}_\perp^2(1-x)^2\alpha(1-\alpha)]^2}\right]$$

Depends on $\vec{\Delta}_\perp$ only through $\vec{\Delta}_\perp(1-x) \Rightarrow$ (as expected!) $\vec{\Delta}_\perp$ dependence disappears as $x \to 1$.

The Physics of $E(x,0,t)$

So far: only unpolarized (or longitudinally) polarized nucleon.

For polarized nucleons, use ($\Delta^+ = 0$)

$$\int \frac{dx^-}{4\pi} e^{ip^+x^-x} \langle P{+}\Delta,\uparrow|\bar{q}(0)\,\gamma^+ q(x^-)|\,P,\uparrow\rangle \;=\; H(x,0,-\boldsymbol{\Delta}_\perp^2)$$

$$\int \frac{dx^-}{4\pi} e^{ip^+x^-x} \langle P{+}\Delta,\uparrow|\bar{q}(0)\,\gamma^+ q(x^-)|\,P,\downarrow\rangle \;=\; -\frac{\Delta_x - i\Delta_y}{2M} E(x,0,-\boldsymbol{\Delta}_\perp^2)$$

$\hookrightarrow$ GPD for nucleon polarized in the x direction (in the IMF) reads

$$F_q(x,0,-\boldsymbol{\Delta}_\perp^2) = H(x,0,-\boldsymbol{\Delta}_\perp^2) + i\frac{\Delta_y}{2M}E(x,0,-\boldsymbol{\Delta}_\perp^2)$$

$\hookrightarrow$ (unpolarized) parton distribution in the $\perp$ plane for a nucleon that is polarized in the x direction given by

$$q_x(x,\mathbf{b}_\perp) = \int \frac{d^2\boldsymbol{\Delta}_\perp}{(2\pi)^2}\left[H(x,-\boldsymbol{\Delta}_\perp^2) + i\frac{\Delta_y}{2M}E(x,-\boldsymbol{\Delta}_\perp^2)\right]e^{-\mathbf{b}_\perp\cdot\boldsymbol{\Delta}_\perp}$$

$\hookrightarrow$ $\frac{\Delta_\perp}{M}E(x,0,-\boldsymbol{\Delta}_\perp^2)$ **describes how the momentum distribution of unpolarized partons in the $\perp$ plane depends on the polarization of the nucleon**.

- positivity constraint for FT of $E(x,0,t)$:

$$\left|\frac{\nabla_{b_\perp}}{2M}\int d^2\mathbf{b}_\perp e^{i\mathbf{b}_\perp\cdot\boldsymbol{\Delta}_\perp}E(x,0,-\boldsymbol{\Delta}_\perp^2)\right| < \int d^2\mathbf{b}_\perp e^{i\mathbf{b}_\perp\cdot\boldsymbol{\Delta}_\perp}H(x,0,-\boldsymbol{\Delta}_\perp^2)$$

3. physical interpretation of Ji's angular momentum sum rule

$$\langle J_q \rangle = \frac{1}{2} \int dx x \left[H_q(x,0,0) + E_q(x,0,0) \right]$$

.

Physics: GPDs (for $\xi = 0$) allow the simultaneous determination of the momentum of partons in the z direction and their position in the $\perp$ direction (compare angular momentum $L_x = yp_z - zp_y$)
$\hookrightarrow$ not surprising to find $GPDs \Leftrightarrow J_q$.

consider nucleons polarized in the x-direction (in rest frame $\rightarrow$ include Melosh rotation!)

$$\begin{aligned} F_q(x,0,\mathbf{\Delta}_\perp) &= H(x,0,-\mathbf{\Delta}_\perp^2) \\ &+ i\frac{\Delta_y}{2M}\left[H(x,0,-\mathbf{\Delta}_\perp^2)+E(x,0,-\mathbf{\Delta}_\perp^2)\right] \end{aligned}$$

Take

- First moment w.r.t. x (to get p_z
- derivative w.r.t. Δ_y (to get y)

$\hookrightarrow$

$$\langle J_q\rangle = \frac{1}{2}\int dx x\left[H_q(x,0,0)+E_q(x,0,0)\right]$$

.

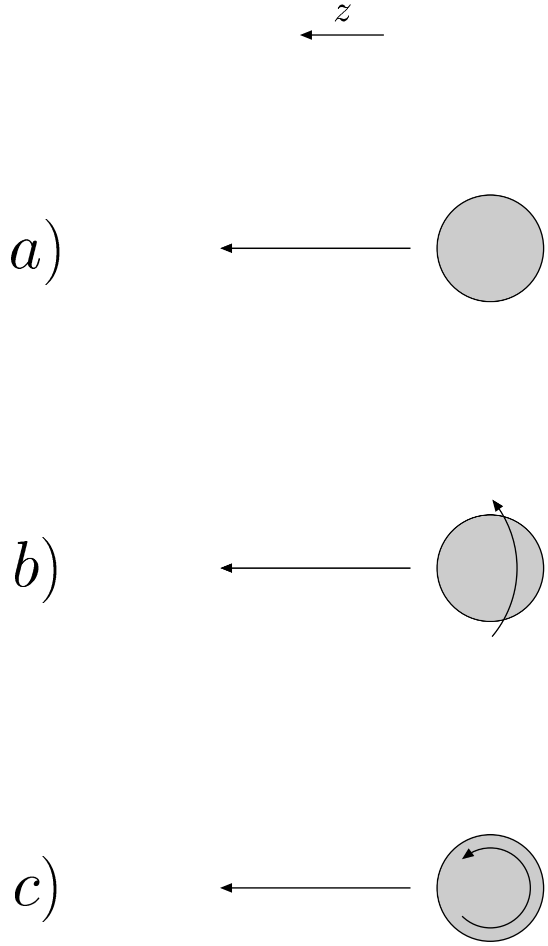


Figure 1: Comparison of a) a non-rotating sphere that moves in z direction with b) sphere that spins at the same time around the z axis and c) sphere that spins around the x axis

When the sphere spins around the x axis, the rotation changes the distribution of momenta in the z direction (adds/subtracts to velocity for $y > 0$ and $y < 0$ respectively)

For the nucleon the analogous modification is described by $E(x, t)$.

summary

- DVCS allows probing generalized parton distributions

$$\int \frac{dx^-}{2\pi} e^{ixp^+x^-} \left\langle p' \left| \bar{\psi}\left(-\frac{x^-}{2}\right) \gamma^+ \psi\left(\frac{x^-}{2}\right) \right| p \right\rangle$$

  GPDs defined through matrix elements of light-cone correlation (similar to usual parton distributions), but $q \equiv p' - p \neq 0$.

- GPDs resemble both usual parton distributions and form factors.

- t-dependence of $\xi = 0$ GPDs (i.e. only $\perp$ momentum transfer) can be interpreted as Fourier transform of impact parameter dependent parton distributions $q(x, \vec{b}_\perp)$

$$\boxed{\begin{array}{l} H(x, 0, -\vec{\Delta}_\perp^2) = \int d^2 b_\perp q(x, \vec{b}_\perp) e^{-i\vec{b}_\perp \vec{\Delta}_\perp} \\ \tilde{H}(x, 0, -\vec{\Delta}_\perp^2) = \int d^2 b_\perp \Delta q(x, \vec{b}_\perp) e^{-i\vec{b}_\perp \vec{\Delta}_\perp} \end{array}}$$

- $q(x, \mathbf{b}_\perp)$, $\tilde{q}(x, \mathbf{b}_\perp)$ have density interpretation (e.g. $q(x, \mathbf{b}_\perp) > 0$ for $x > 0$ and $\int d^2\mathbf{b}_\perp q(x, \mathbf{b}_\perp) = q(x)$

$\hookrightarrow$ knowledge of GPDs for $\xi = 0$ allows determining distribution of partons in the $\perp$ plane (as function of distance to $\perp$ center of momentum)

$\hookrightarrow$ provides completely new information about parton structure of nucleons!

$\hookrightarrow$ novel probe for nonperturbative parton physics

- universal prediction: large x partons more localized in $\mathbf{b}_\perp$ than small x partons
- correlate with other experiments that are sensitive to distribution of partons in $\perp$ plane, such as multiple parton scattering, ...
- DVCS experiments only probe $\xi \neq 0$, but extrapolation to $\xi = 0$ possible since moments of GPDs have polynomial ξ dependence.
- published in: M.B., PRD **62**, 71503 (2000); see also: hep-ph/0008051 and hep-ph/0010082.