On the Way to QCD Precision Test with Deep Inelastic Scattering

Johannes Blümlein

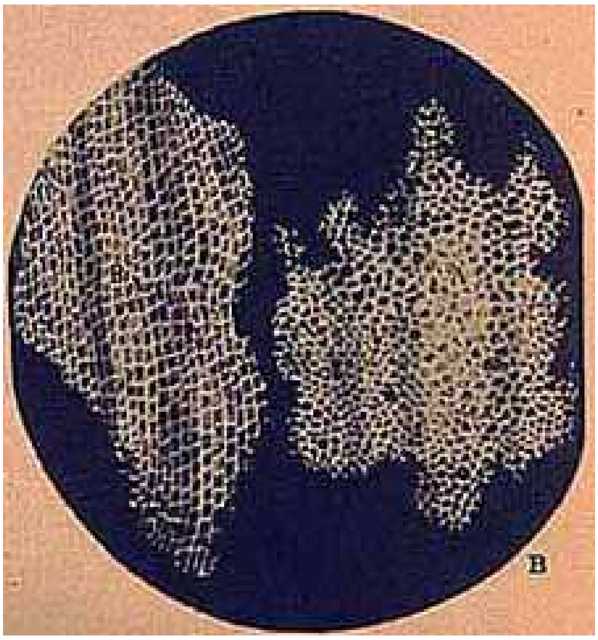
DESY

1. Introduction

The Door to the Very Small is Opened by Microscopes.

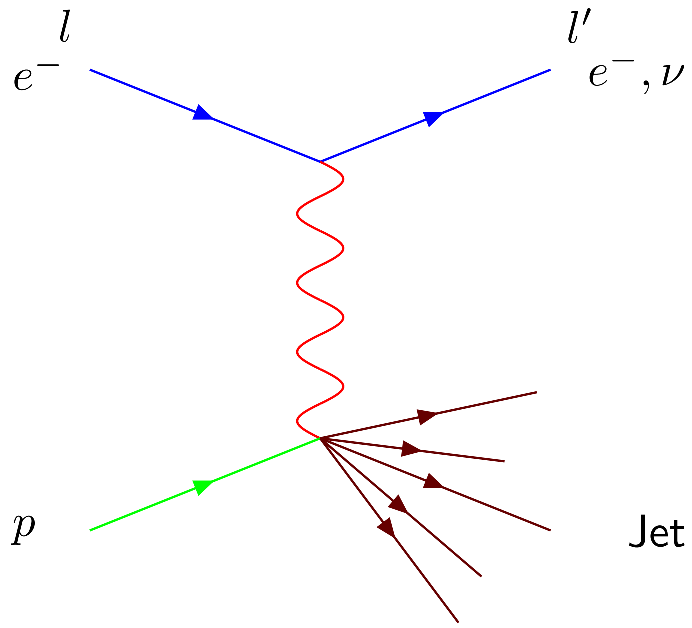
Robert Hooke (1635-1703)

Remake of the original microscope



Observation of cork cells

DEEPLY INELASTIC SCATTERING



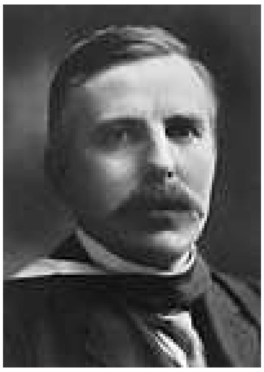
space-like process :

$$
\begin{aligned}
q^2 &= (l - l')^2 = -Q^2 < 0 \\
W^2 &= (p + q)^2 \geq M_p^2
\end{aligned}
$$

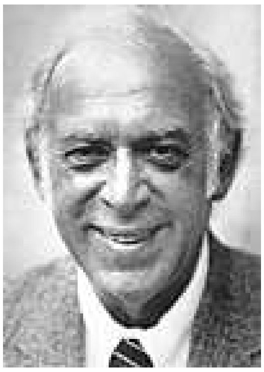
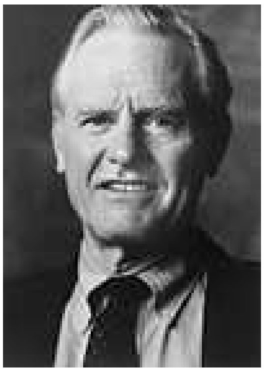
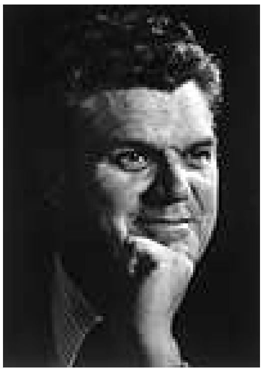
$$x = \frac{Q^2}{2p.q}, \qquad y = \frac{p.q}{p.l}$$

$$0 \leq x, y \leq 1$$

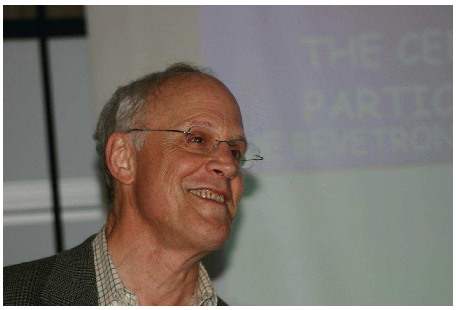
Study of the Nucleon Structure

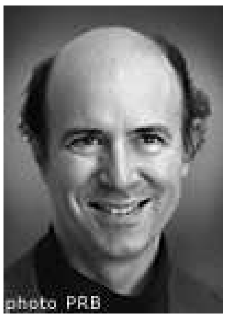



Rutherford Chadwick Stern Hofstadter


Friedman Kendall Taylor Bjorken
Dirac Medal 2004

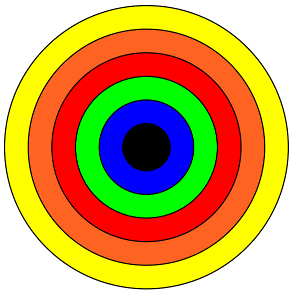
Feynman Gross (LL2004: April DESY) Politzer Wilczek
Nobel Laureates 2004

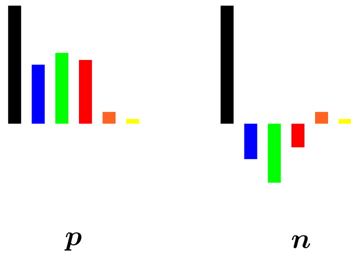
The Resolution of the Nucleon Microscope

$$\Delta x \sim \frac{1}{|Q|} = \frac{1}{\sqrt{-q^2}}$$

Examples :

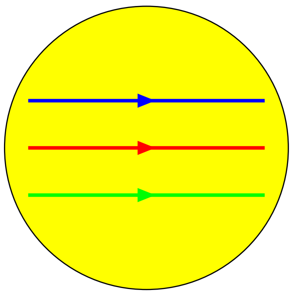
$Q^2 \sim 0.5 \cdot M_p^2$





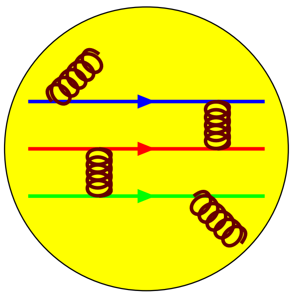
Charge Distribution

$Q^2 \sim 3 \cdot M_p^2$



Scaling

$Q^2 \sim 10...500 \cdot M_p^2$



Violation of Scaling

If there are New Compositeness Scales, one may find them in the Future.

$$Q^2 > 10^4 \,\text{GeV}^2, \qquad 1 \,\text{GeV}^2 \sim M_p^2$$

When is a Parton ?

S. DRELL: **Infinite Momentum Frame:** P - large

$$\tau_{\rm int} \ll \tau_{\rm life}$$

$$\tau_{\rm int} \sim \frac{1}{q_0} = \frac{4Px}{Q^2(1-x)}$$

$$\tau_{\rm life} \sim \frac{1}{\sum_i E_i - E} = \frac{2P}{\sum_i (k_{\perp i}^2 + M_i^2)/x_i - M^2} \simeq \frac{2Px(1-x)}{k_\perp^2}$$

$$\frac{\tau_{\rm int}}{\tau_{\rm life}} = \frac{2k_\perp^2}{Q^2(1-x)^2}$$

Stay away from $x \rightarrow 0$, since xP becomes too small.

Stay away from $x \rightarrow 1$.

$$Q^2 \gg k_\perp^2.$$

Main Research Objectives :

- ☞ Precise Measurement of $\alpha_s(M_Z^2)$
- ☞ Reveal polarized and unpolarized parton densities at highest precision
- ☞ Precision tests of QCD
- ☞ Find novel sub-structures

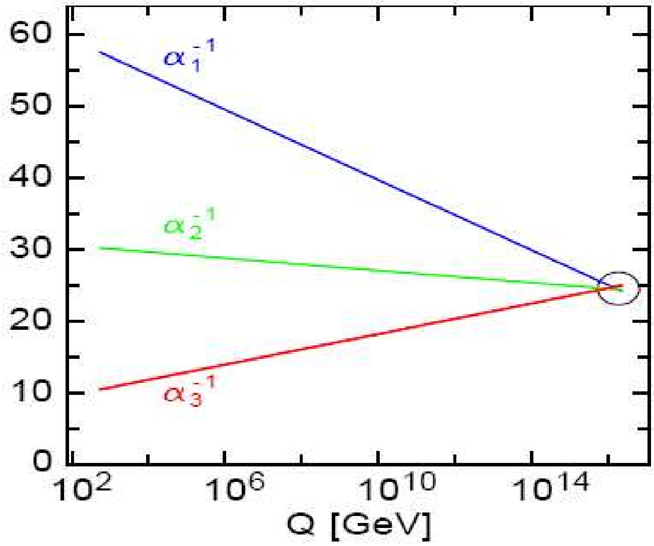
$\Longrightarrow$ Perturbative QCD :

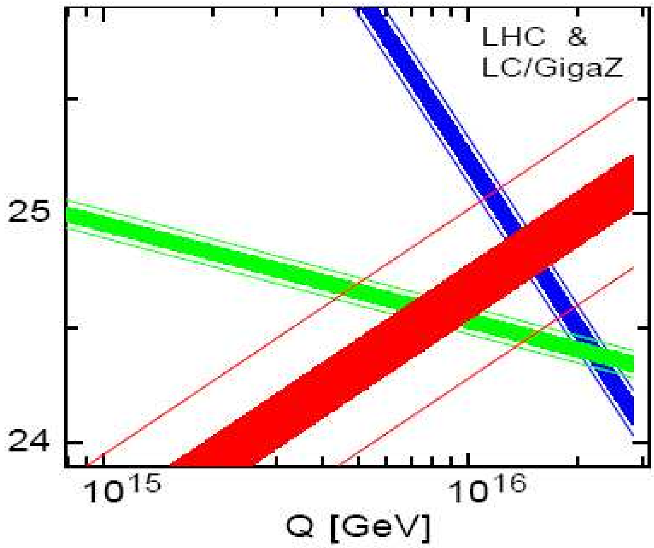
NNLO calculations using new technologies

$\Longrightarrow$ Lattice QCD :

Calculation of certain non-perturbative quantities a priori

Unification of Forces and α_s





P. Zerwas, 2004

$$\frac{\delta\alpha(0)}{\alpha(0)} \sim 3 \cdot 10^{-11} \qquad \frac{\delta\alpha_w}{\alpha_w} \sim 7 \cdot 10^{-4} \qquad \frac{\delta\alpha_s(M_Z^2)}{\alpha_s(M_Z^2)} \sim 2 \cdot 10^{-2}$$

2. Basic Techniques

$$\frac{d\sigma^{\rm DIS}}{dxdy} \propto \sum_{s'} \overline{|M|^2} = \frac{1}{Q^4} \quad L_{\mu\nu} \quad W^{\mu\nu}, \qquad \text{pure} \;\; \gamma \;\; \text{exchange.}$$

$L_{\mu\nu}$	$-$	calculable
$W^{\mu\nu}$	$-$	not calculable

Parameterize: according to the symmetries *P, T, C, etc.*

$$W^{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x,Q^2) + \frac{1}{M_p^2} \widehat{P}_\mu \widehat{P}_\nu W_2(x,Q^2) + \dots \; .$$

$$\widehat{P}_\mu = p_\mu - \frac{q.p}{q^2} q_\mu \; .$$

The Parton Model :

R.P. Feynman, 1969; J.D. Bjorken, E.A. Paschos, 1969

ANSATZ:

$W_i(x, Q^2)$ **is obtained as an integral over the momentum distributions of** Local Sub-Components, the Partons.

$$W_2(x, Q^2) = \sum_i \int_0^1 dx_i f(x_i) x_i e_i^2 \delta\left(\frac{q.p_i}{M^2} - \frac{Q^2}{2M}\right)$$

$\Longrightarrow$ Strong correlation between $p.q$ and Q^2

$\Longrightarrow$ "Micro Canonical Ensemble"

$f_i(x)$ - Distribution Function

$$q.p_i = x_i p.q, \quad 2p.q = Q^2/x, \quad M\nu = p.q$$

$$\nu W_2(x, Q^2) = \sum_i e_i^2 x f_i(x) \equiv F_2(x) \ .$$

Bjorken Limit :

$$Q^2 \to \infty, \qquad \nu \to \infty$$

$$x = \text{const.}$$

Scaling :

$$MW_1(\nu, Q^2) \to F_1(x)$$
$$\nu W_2(\nu, Q^2) \to F_2(x)$$

The Light Cone Expansion :

More general approach, allowing for higher twist.

Brandt, Preparata, Zimmermann, Frishman, Christ et al.

$$W_{\mu\nu}(p.q) = \int d^4 x e^{iqx} \langle p \, | [j_\mu(x), j_\nu(0)] | \, p \rangle$$

$$\begin{aligned} T\left[j_\mu(x), j_\nu(0)\right] &= \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{\pi^4 (x^2 - i\varepsilon)^4} + O_{\mu\nu} \\ & -i \frac{x^\lambda \sigma_{\mu\lambda\nu\rho} O_V^\rho(x,0)}{2\pi^2(x^2 - i\varepsilon)} - i \frac{x^\lambda \varepsilon_{\mu\lambda\nu\rho} O_{V5}^\rho(x,0)}{2\pi^2(x^2 - i\varepsilon)} \end{aligned}$$

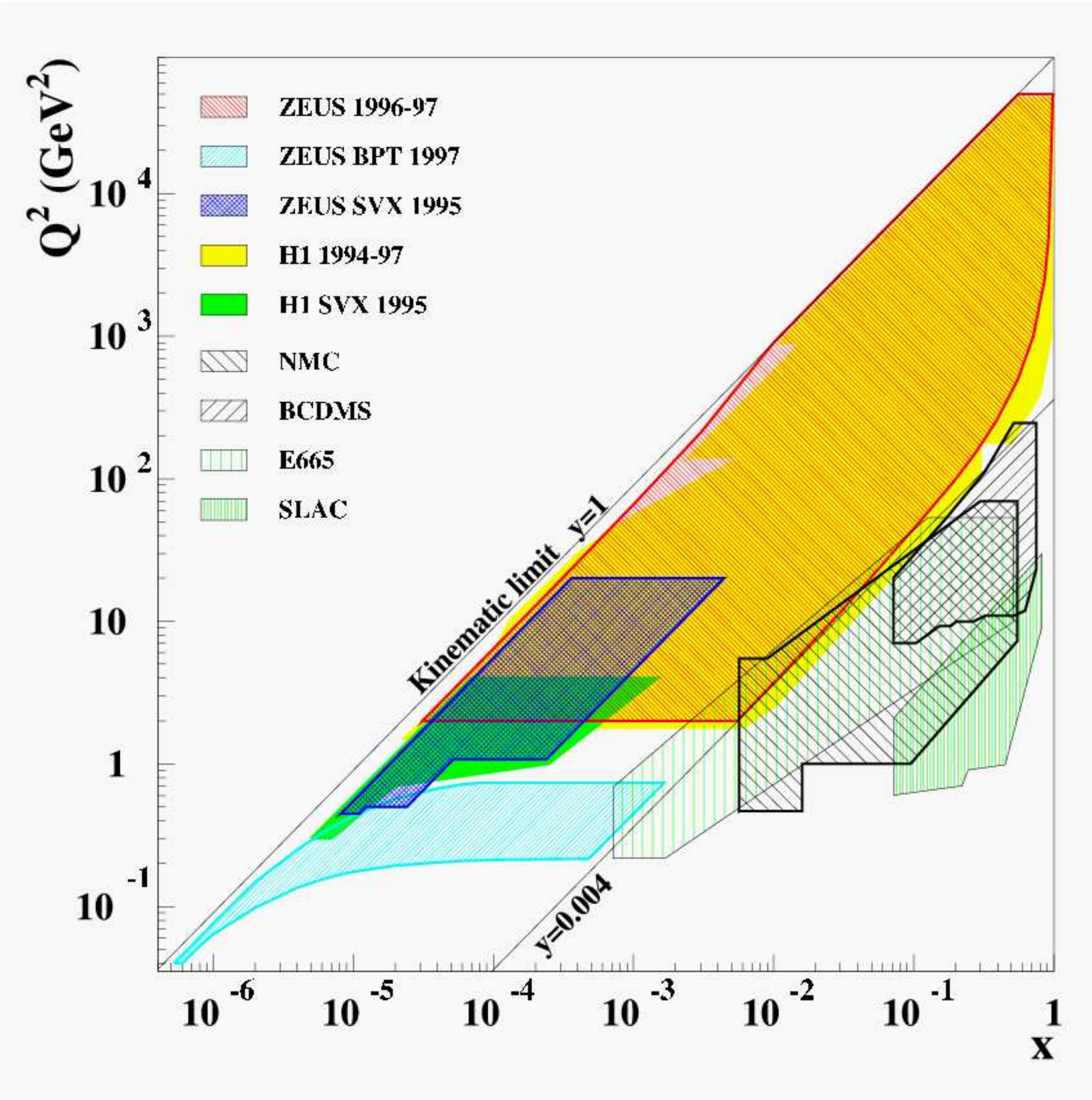
$$\begin{aligned} O_V^\mu(x,y) &= \; : \overline{\psi(x)} \gamma^\mu \psi(y) - \overline{\psi(y)} \gamma^\mu \psi(x) : \\ O_{V5}^\mu(x,y) &= \; : \overline{\psi(x)} \gamma^\mu \gamma_5 \psi(y) - \overline{\psi(y)} \gamma^\mu \gamma_5 \psi(x) : \\ O^{\mu\nu}(x,y) &= \; : \overline{\psi(x)} \gamma^\mu \psi(x) \overline{\psi}(y) \gamma^\nu \psi(x) : \\ \psi(x) &= \psi(0) + x^\mu \left[\partial_\mu \psi(x)\right]_{x=0} + \frac{1}{2!} x^\mu x^\nu \left[\partial_\mu \partial_\nu \psi(x)\right]_{x=0} \\ & + \ldots \\ O_{V,V5}^\mu(x,0) &= \sum_{n=0}^\infty \frac{1}{n!} x^{\mu_1} \ldots x^{\mu_n} O_{V,V5,\mu_1,\ldots,\mu_n}^\mu(0) \end{aligned}$$

$\Longrightarrow$ Calculate anomalous dimensions for Operators.

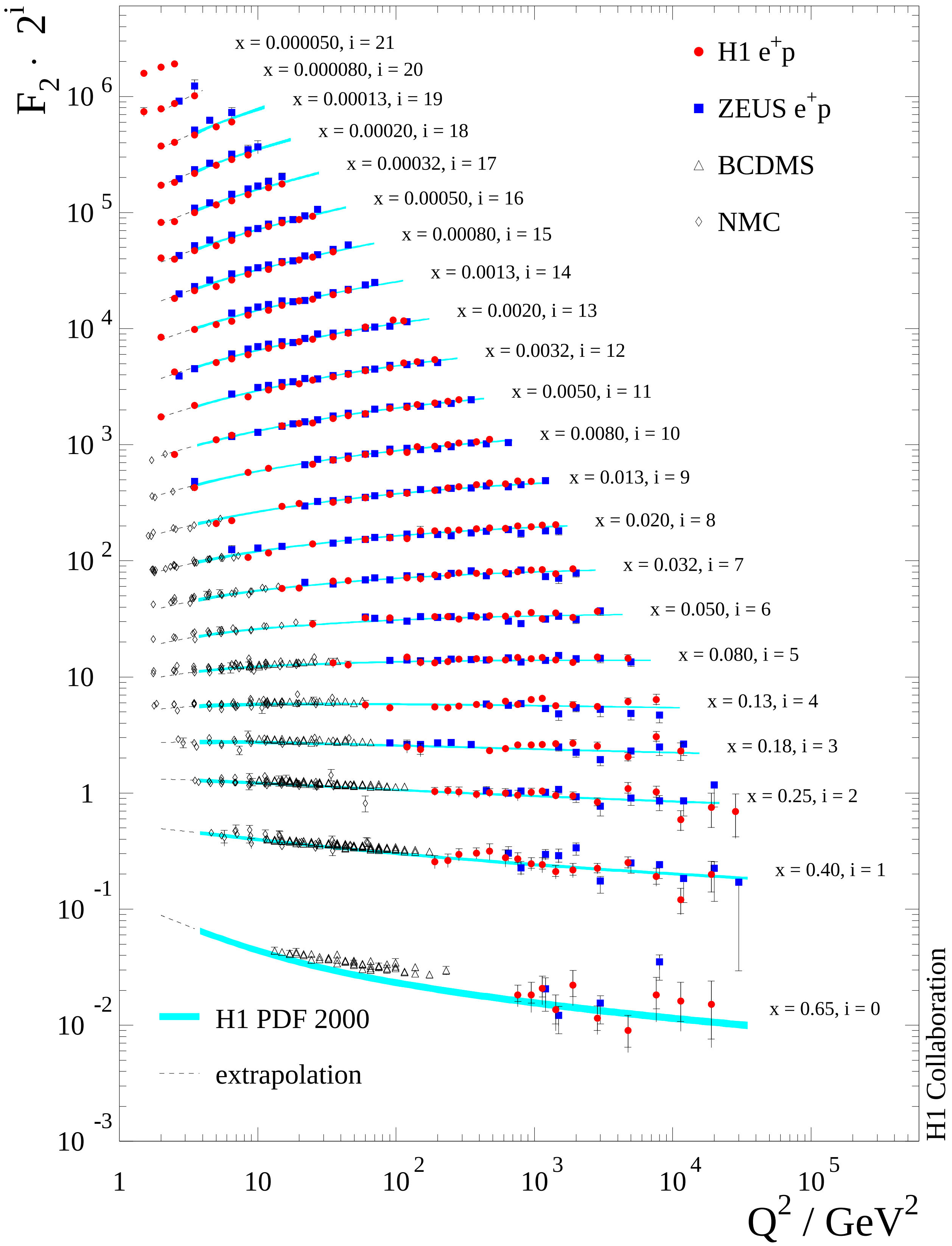
$\Longrightarrow$ Only safe way to Higher Twists

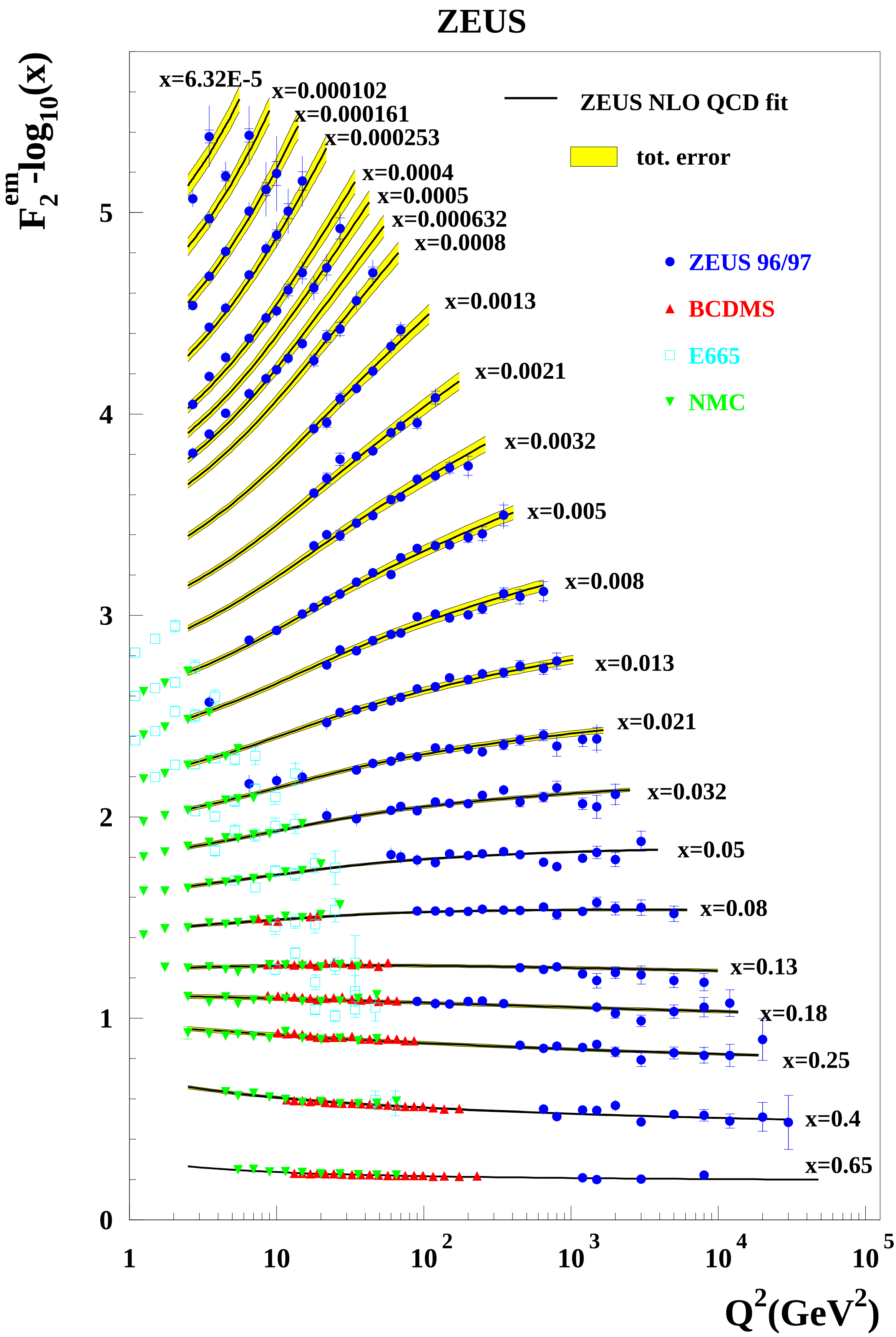
Twist 2: LCE $\simeq$ Parton Model

Kinematic Domain

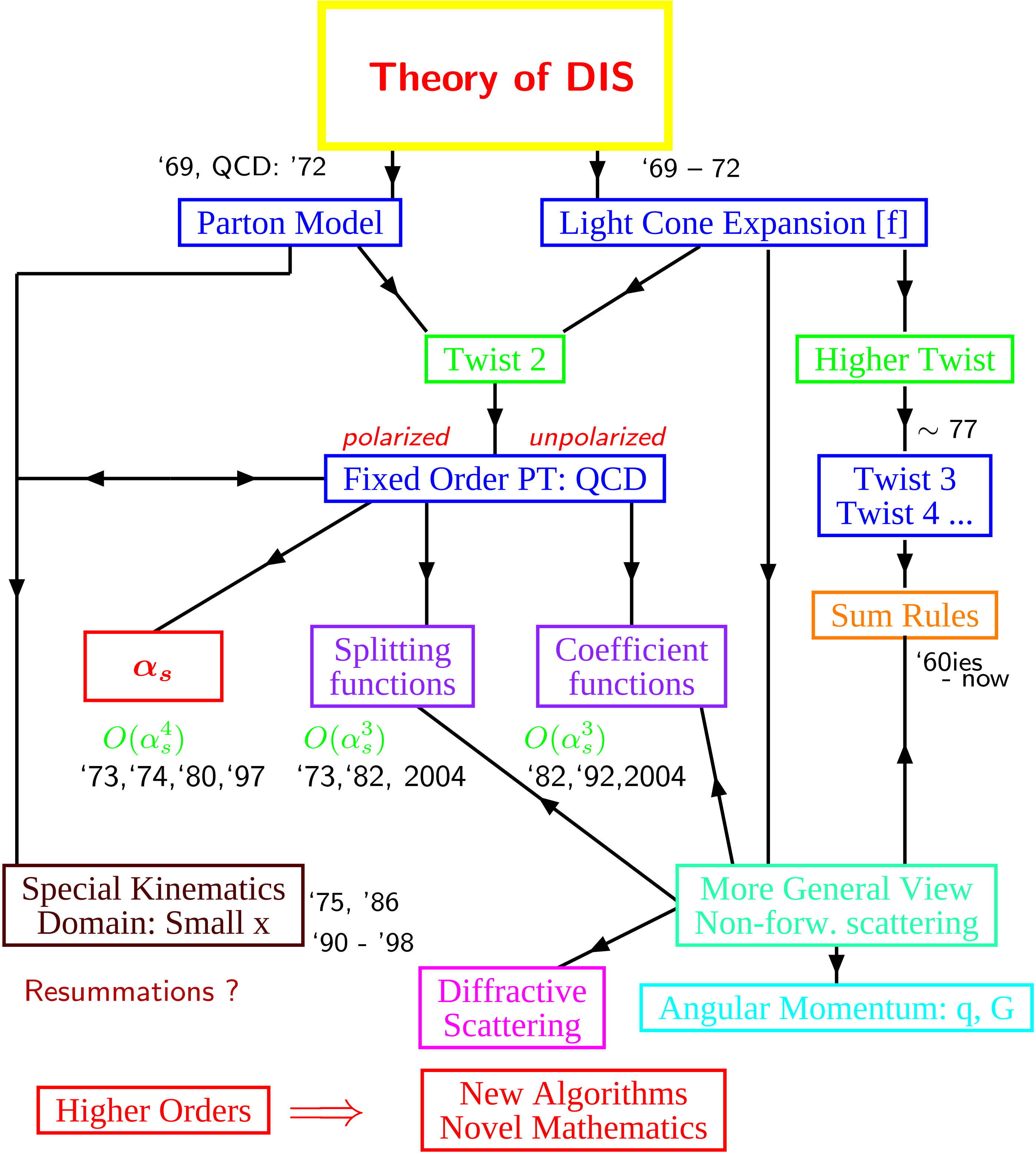


H1, ZEUS + fixed target data





Scaling violations of $F_2(x, Q^2)$.



Theory of DIS
'69, QCD: '72
'69 – 72
Parton Model
Light Cone Expansion [f]
Twist 2
Higher Twist
~ 77
polarized
unpolarized
Fixed Order PT: QCD
Twist 3
Twist 4 ...
Sum Rules
'60ies - now
α_s
Splitting functions
Coefficient functions
$O(\alpha_s^4)$
'73,'74,'80,'97
$O(\alpha_s^3)$
'73,'82, 2004
$O(\alpha_s^3)$
'82,'92,2004
Special Kinematics
Domain: Small x
'75, '86
'90 - '98
More General View
Non-forw. scattering
Resummations ?
Diffractive Scattering
Angular Momentum: q, G
Higher Orders
New Algorithms
Novel Mathematics

3. QCD Perturbation Theory to $O(\alpha_s^3)$, $\Lambda_{\rm QCD}$ and the PDF's

How can we measure $\alpha_s(Q^2)$ from the scaling violations of Structure Functions?

$$
\begin{aligned}
F_j(x,Q^2) &= \hat{f}_i(x,\mu^2) \otimes \sigma_j^i\left(\alpha_s, \frac{Q^2}{\mu^2}, x\right) \\
& \quad\ \uparrow \text{bare pdf} \quad \uparrow \text{sub} - \text{system cross} - \text{sect.} \\
&= \underbrace{\hat{f}_i(x,\mu^2) \otimes \Gamma_k^i\left(\alpha_s(R^2), \frac{M^2}{\mu^2}, \frac{M^2}{R^2}\right)}_{\text{finite pdf} \equiv \text{f}_\text{k}} \\
& \quad \otimes \underbrace{C_j^k\left(\alpha_s(R^2), \frac{Q^2}{\mu^2}, \frac{M^2}{R^2}, x\right)}_{\text{finite Wilson coefficient}}
\end{aligned}
$$

Move to Mellin space :

$$F_j(N) = \int_0^1 dx x^{N-1} F_j(x)$$

Diagonalization of the convolutions $\otimes$ into ordinary products.

Renormalization Group Equations :

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}-2\gamma_\psi(g)\right]F_i(N) \quad = \quad 0$$

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}+\gamma_\kappa^N(g)-2\gamma_\psi(g)\right]f_k(N) \quad = \quad 0$$

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}-\gamma_\kappa^N(g)\right]C_j^k(N) \quad = \quad 0$$

Callan–Symnanzik equations for mass factorization

$\equiv$ Altarelli–Parisi evolution equations

x-space :

$$\frac{d}{d\log(\mu^2)}\begin{pmatrix} q^+(x,Q^2) \\ G(x,Q^2)\end{pmatrix} = \frac{\alpha_s}{2\pi}\boldsymbol{P}(x,\alpha_s)\otimes\begin{pmatrix} q^+(x,Q^2) \\ G(x,Q^2)\end{pmatrix}$$

$$\boldsymbol{P}(x,\alpha_s)=\boldsymbol{P}^{(0)}(x)+\frac{\alpha_s}{2\pi}\boldsymbol{P}^{(1)}(x)+\left(\frac{\alpha_s}{2\pi}\right)^2\boldsymbol{P}^{(2)}(x)+\dots$$

Evolution equs.: 3 non-singlet, 1 singlet

Separation of non-singlet and singlet quark contributions is **essential.**

3.1. Running Coupling Constant

$$\frac{\partial a_s(\mu^2)}{\partial \log \mu^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6)$$

$$a_s \equiv \frac{g_{\rm ren}^2}{(4\pi)^2} = \frac{\alpha_s}{2\pi}$$

The values of the β_k :

$$\beta_0 = 11 - \frac{2}{3} N_f$$ GROSS, POLITZER, WILCZEK, T'HOOFT, 1973

DISCOVERY OF ASYMPTOTIC FREEDOM :

NOBEL LAUREATES 2004

$$\beta_1 = 102 - \frac{38}{3} N_f$$ CASWELL(†11.9.01), JONES, 1974

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2$$

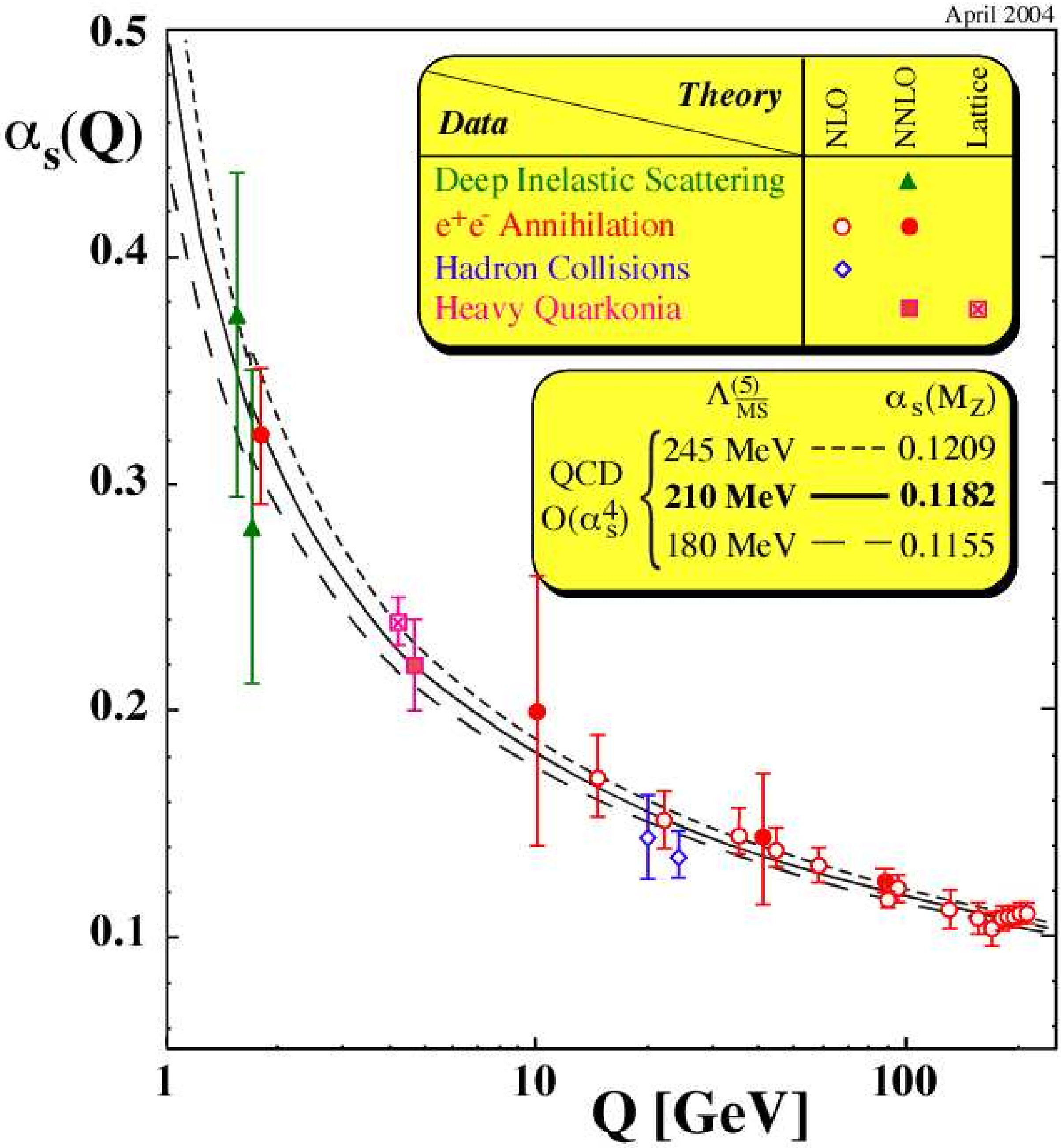
TARASOV, VLADIMIROV, ZHARKOV, 1981

LARIN, VERMASEREN, 1992

$$\beta_3 = \left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right) N_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) N_f^2 + \frac{1093}{729} N_f^3$$

VAN RITBERGEN, VERMASEREN, LARIN, 1997

THE SOLUTION OF THE RGE LEADS TO A FALLING COUPLING CONSTANT AS SCALES INCREASE.



S. Bethke, LL2004.

3.2. Splitting Functions

$O(\alpha_s)$ unpolarized:

$$
\begin{aligned}
P_{\rm NS}^{(0)}(z) \equiv P_{qq}^{(0)}(z) &= C_F \left[\frac{1+z^2}{1-z}\right]_+ \\
P_{qg}^{(0)}(z) &= T_f \left[(1-z)^2 + z^2\right] \\
P_{gq}^{(0)}(z) &= C_F \frac{1+(1-z)^2}{z} \\
P_{gg}^{(0)}(z) &= 2C_A \left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\right] + \frac{1}{2}\beta_0 \delta(1-z)
\end{aligned}
$$

QED : P_{qq} Fermi, 1924 P_{gq} Williams, 1933; Weizsäcker, 1934

Gross, Wilczek; Georgi, Politzer, 1973;

further: Lipatov, 1975; Altarelli, Parisi, 1977; Kim, Schilcher, 1977; Dokshitser, 1977

$O(\alpha_s)$ polarized:

$$
\begin{aligned}
\Delta P_{qq}^{(0)}(z) &= P_{qq}^{(0)}(z) \\
\Delta P_{qg}^{(0)}(z) &= T_f \left[(1-z)^2 - z^2\right] \\
\Delta P_{gq}^{(0)}(z) &= C_F \frac{1-(1-z)^2}{z} \\
\Delta P_{gg}^{(0)}(z) &= 2C_A \left[\left(\frac{1}{1-z}\right)_+ + 1 - 2z\right] + \frac{1}{2}\beta_0 \delta(1-z)
\end{aligned}
$$

Ito, 1975; K. Sasaki, 1975; Ahmed & Ross 1975,1976;

correct: Altarelli, Parisi, 1977.

no terms $\propto 1/z$.

2 Loop :

unpolarized:

Floratos, D. Ross, Sachraida, 1977-79; Curci, Furmanski, Pertonzio, 1980; Furmanski, Petronzio, 1980; Gonzalez-Arroyo, Lopez, Ynduraın, 1979, 1980; Floratos, Kounnas, Lacaze, 1981abc; van Neerven, Hamberg, 1982;

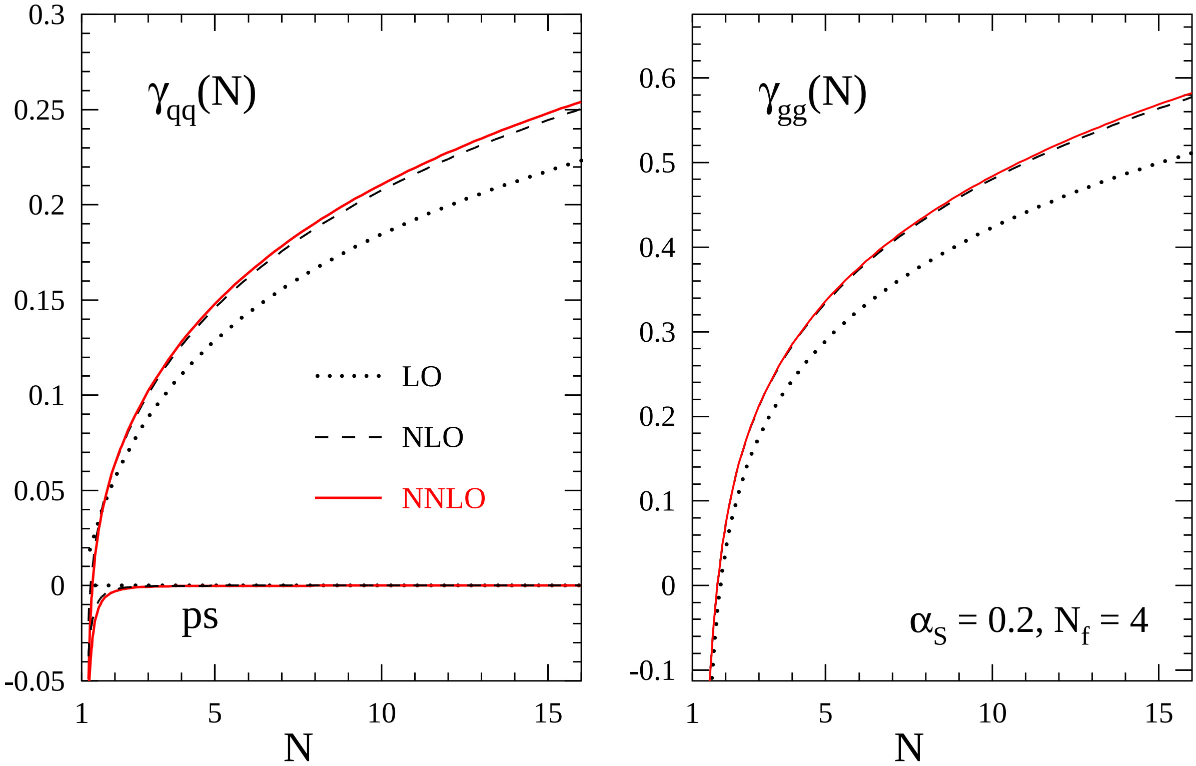
polarized:

Zijlstra, van Neerven, 1994; Mertig, van Neerven, 1995; Vogelsang 1995.

3 Loop :

unpolarized:

Moments : Larin, Nogueira, van Ritbergen, Vermaseren, 1994, 1997; Retey, Vermaseren, 2001; J.B., Vermaseren, 2004.
Complete : Moch, Vermaseren, Vogt, 2004.



3.3. Coefficient Functions

$O(\alpha_s)$ unpolarized:

$$
\begin{aligned}
C^{(1)}_{F_2^q}(z) &= C_F \left\{ \frac{1+z^2}{1-z} \left[\ln\left(\frac{1-z}{z}\right) - \frac{3}{4} \right] + \frac{1}{4}(9+5z) \right\}_+ \\
C^{(1)}_{F_2^g}(z) &= 2N_f T_f \left\{ \left[z^2 + (1-z)^2\right] \ln\left(\frac{1-z}{z}\right) - 1 + 8z(1-z) \right\} \\
C^{(1)}_{F_1^q}(z) &= C^{(1)}_{F_2^q}(z) - C_F \cdot 2z \\
C^{(1)}_{F_1^g}(z) &= C^{(1)}_{F_2^g}(z) - 8 N_f T_f z(1-z) \\
C^{(1)}_{F_3^q}(z) &= C^{(1)}_{F_2^q}(z) - C_F(1+z)
\end{aligned}
$$

FURMANSKI, PETRONZIO, 1982: correct form.

$O(\alpha_s)$ polarized:

$$
\begin{aligned}
C^{(1)}_{g_1^q}(z) &= C^{(1)}_{F_1^q}(z) \\
C^{(1)}_{g_1^g}(z) &= 4 N_f T_f \left\{ [2z-1] \ln\left(\frac{1-z}{z}\right) + 3 - 4z \right\}
\end{aligned}
$$

ALTARELLI, ELLIS, MARTINELLI, 1979; HUMPERT, VAN NEERVEN, 1981; BODWIN QUI, 1990.

2 Loop :

polarized, unpolarized:

Zijlstra, van Neerven 1992–1994;
Moments: Moch, Vermaseren, 1999

unpolarized, heavy flavor:

Laenen, Riemersma, Smith, van Neerven, 1993, 1994
Mellin Space: Alekhin, J.B., 2004

3 Loop :

unpolarized:

Moments : Larin, Nogueira, van Ritbergen, Vermaseren, 1994, 1997; Retey, Vermaseren, 2001; J.B., Vermaseren, 2004.
Complete : Moch, Vermaseren, Vogt, in preparation.

Example : J.B., Vermaseren, 2004

$$
\begin{aligned}
C_2^{\mathrm{NS},16}(a_s) \quad = \quad & \frac{4047739719}{190590400} C_F\, a_s \\
+ \quad & \Bigg[\Bigg(\frac{4442667416304442887936697 0127}{32193184692174795646156 8000} \frac{24439538}{255255} \zeta_3 \Bigg) {C_F}^2 \\
+ \quad & \Bigg(\frac{1791830840849829422278 3087}{594227058731828121600 00} - \frac{113298677}{1021020} \zeta_3 \Bigg) C_F\, C_A \\
- \quad & \frac{1435683727619074721111 77}{275891134411205913600 0} C_F\, N_F \Bigg] {a_s}^2 \\
+ \quad & \Bigg[\Bigg(\frac{59290512768143}{3127445521200} \zeta_4 - \frac{27643576}{21879} \zeta_5 \\
+ \quad & \frac{303681339759950972508467729384250597655 9161689}{803445801604077593342164786340334796800 0000} \\
+ \quad & \frac{149434192694045086538 7403}{595674040206012768000} \zeta_3 \Bigg) {C_F}^3 \\
+ \quad & \Bigg(\frac{59290512768143}{6254891042400} \zeta_4 + \frac{26286537788347572655880093551503319 0333}{56646805852503848671021043712000000} \\
+ \quad & \frac{47187263}{51051} \zeta_5 - \frac{1535505046917148231 3}{4991403051835200} \zeta_3 \Bigg) C_F\, {C_A}^2 \\
+ \quad & \Bigg(\frac{72273849359996703123187898 84999}{76056398835262954714045440000} + \frac{64419601}{20675655} \zeta_3 \Bigg) C_F\, {N_F}^2 \\
+ \quad & \Bigg(\frac{7750026627118768752845091760890051465242741}{16525006203292422734310258871664640 00000} \\
- \quad & \frac{28494820041389214915 31}{6741167121672984000} \zeta_3 + \frac{983963}{21879} \zeta_5 \\
- \quad & \frac{59290512768143}{2084963680800} \zeta_4 \Bigg) {C_F}^2 C_A + \Bigg(- \frac{552298563960959}{4021001384400} \zeta_3 \\
- \quad & \frac{407320724134849319615222207993355 7529}{352977746994455372827884887040 0000} + \frac{64419601}{1531530} \zeta_4 \Bigg) {C_F}^2 N_F \\
+ \quad & \Bigg(\frac{598788865585667}{1850495446800} \zeta_3 - \frac{64419601}{1531530} \zeta_4 \\
- \quad & \frac{582811634921542995647179358698536547}{40462004180359891907872174080 0000} \Bigg) C_F\, C_A\, N_F \Bigg] {a_s}^3
\end{aligned}
$$

Agreement with : an upcoming paper by Moch, Vermaseren, Vogt

4. New Mathematics in Perturbation Theory

Consider hard scattering processes in massless field theories:

QCD, QED, $m_i \to 0$

Factorization Theorem Leading Twist:

The cross section σ factorizes as

$$\sigma = \sum_k \sigma_{k,W} \otimes f_k$$

σ_W perturbative Wilson Coefficient

f non-perturbative Parton Density

$\otimes$ Mellin convolution

$$\begin{aligned} [A \otimes B](x) &= \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2) \\ \mathbf{M}[A \otimes B](N) &= \mathbf{M}[A](N) \cdot \mathbf{M}[B](N) \end{aligned}$$

with the Mellin transform :

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x), \qquad Re[N] > c$$

Observation :

Feynman Amplitudes seem to obey the **Mellin Symmetry**

i.e. to significantly simplify in **Mellin Space**

van Neerven, Zijlstra 1992

$$
\begin{aligned}
&c_{2,-}^{(2)}(x) = C_F\,(C_F - C_A/2) \times \\
&\left\{ \frac{1+x^2}{1-x} \left[\left[4\ln^2(x) - 16\ln(x)\ln(1+x) - 16\mathrm{Li}_2(-x) - 8\zeta_2 \right] \ln(1-x) \right.\right. \\
&+ \left[-2\ln^2(x) + 20\ln(x)\ln(1+x) - 8\ln^2(1+x) + 8\mathrm{Li}_2(1-x) + 16\mathrm{Li}_2(-x) - 8 \right] \ln(x) \\
&-16\ln(1+x)\mathrm{Li}_2(-x) - 8\zeta_2\ln(1+x) - 16\left[\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) - \mathrm{Li}_3\left(\frac{1-x}{1+x}\right) \right] \\
&\left. -16\mathrm{Li}_2(1-x) + 8S_{1,2}(1-x) + 8\mathrm{Li}_3(-x) - 16S_{1,2}(-x) + 8\zeta_3 \right] \\
&+(4+20x)\left[\ln^2(x)\ln(1+x) - 2\ln(x)\ln^2(1+x) - 2\zeta_2\ln(1+x) - 4\ln(1+x)\mathrm{Li}_2(-x) \right. \\
&\left. +2\mathrm{Li}_3(-x) - 4S_{1,2}(-x) + 2\zeta_3 \right] + \left(32 + 32x + 48x^2 - \frac{72}{5}x^3 + \frac{8}{5x^2} \right) \\
&\times \left[\mathrm{Li}_2(-x) + ln(x)\ln(1+x) \right] + 8(1+x)\left[\mathrm{Li}_s(1-x) + \ln(x)\ln(1-x) \right] + 16(1-x)\ln(1-x) \\
&+ \left(-4 - 16x - 24x^2 + \frac{36}{5}x^3 \right) \ln^2(x) + \frac{1}{5}\left(-26 - 106x + 72x^2 - \frac{8}{x} \right) \ln(x) \\
&\left. + \left(-4 + 20x + 48x^2 - \frac{72}{5}x^3 \right) \zeta_2 + \frac{1}{5}\left(-162 + 82x + 72x^2 + \frac{8}{x} \right) \right\}
\end{aligned}
$$

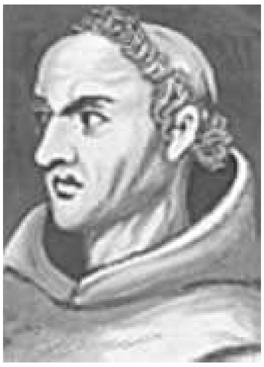
.... several other pages for $c_2^{(+)}(x), c_2^G(x), c_L^{(q,G)}(x)$

$\Longrightarrow$ 77 Functions @ 2 Loops

- $\Longrightarrow$ partly rather complicated arguments
- $\Longrightarrow$ relations are not directly visible ...

The 77 functions do roughly correspond in number to the number of all possible harmonic sums up to weight w=4: 80.

Goal: Simplicity



W. of Occam

Multiple Harmonic Sums to Level 6 :

The simplest example :

$$P_{qq}(x) = \left(\frac{1+x^2}{1-x}\right)_+ = \frac{2}{(1-x)_+} + \ldots$$

$$\int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = -\sum_{k=0}^{N-2} \int_0^1 dx x^k = -\sum_{k=1}^{N-1} \frac{1}{k} = -S_1(N-1)$$

Alternating sums :

$$S_{-1}(N-1) = (-1)^{N-1}\mathbf{M}\left[\frac{1}{1+x}\right](N) - \ln(2) = \int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = \sum_{k=1}^{N-1} \frac{(-1)^k}{k}$$

(Finite for $N \rightarrow \infty$.)

General case :

$$S_{a_1,\ldots,a_l}(N) = \sum_{k_1=1}^{N} \frac{(\mathrm{sign}(a_1))^{k_1}}{k_1^{|a_1|}} \sum_{k_2=1}^{k_1} \frac{(\mathrm{sign}(a_2))^{k_2}}{k_2^{|a_2|}} \ldots$$

Vermaseren, 1997

All Mellin transforms occurring in massless Field Theories for 1-Parameter Quantities can be represented by Harmonic Sums

(at least to 3–loop order).

Algebraic Relations

First relation: L. Euler, 1775

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m+n}, \qquad m, n > 0$$

Generalized to alternating sums by

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m \wedge n}, m \wedge n = [|m| + |n|]\, \mathrm{sign}(m)\mathrm{sign}(n)$$

Ternary relations: Sita Ramachandra Rao, 1984; 4-ary relation: J.B., Kurth, 1998.

These & other relations hold widely independent of their **Value** and **Type**.

Determined by :
- Index Structure
- Multiplication Relation

Ramanujan: integer sums

Faa di Bruno: roots of multivar. algebraic equations

The Formalism applies as well to the Harmonic Polylogarithms. Remiddi, Vermaseren, 1999.

Linear Representations of Mellin Transform by Harmonic Sums:

$$\mathbf{M}[F_w(x)](N) = S^w_{k_1,\ldots,k_m}(N) + P\left(S^{\tau'}_{k_1,\ldots,k_r}, \sigma^{\tau''}_{k_1,\ldots,k_p}\right)$$

$w = \sum_{i=1}^m |k_i|$ Weight

$\tau', \tau'' < w$ P is a polynomial.

w	#	Σ	
1	2	2	
2	6	8	
3	18	26	2 Loop anom. Dimensions
4	54	80	2 Loop Wilson Coefficients
5	162	242	3 Loop anom. Dimensions
6	486	728	3 Loop Wilson Coefficients
	$2 \cdot 3^{w-1}$	$3^w - 1$	

Shuffle Products

Depth 2:

$$S_{a_1}(N) ⧢ S_{a_2}(N) = S_{a_1,a_2}(N) + S_{a_2,a_1}(N)$$

Depth 3:

$$S_{a_1}(N) ⧢ S_{a_2,a_3}(N) = S_{a_1,a_2,a_3}(N) + S_{a_2,a_1,a_3}(N) + S_{a_2,a_3,a_1}(N)$$

Depth 4: .....

Algebraic Equations

Depth 2:

$$S_{a_1}(N) ⧢ S_{a_2}(N) - S_{a_1}(N)S_{a_2}(N) - S_{a_1 \wedge a_2}(N) = 0$$

Depth 3:

$$S_{a_1}(N) ⧢ S_{a_2,a_3}(N) - S_{a_1}(N)S_{a_2,a_3}(N) - S_{a_1 \wedge a_2,a_3}(N) - S_{a_2,a_1 \wedge a_3}(N) = 0$$

Depth 4: ....

Basic Sums = # Permutations - # Independent Equations

Theory of Words

Can we count the Basis in simpler way ? $\Longrightarrow$ YES.

Free Algebras and Elements of the Theory of Codes
$\Longrightarrow$ **Particle Physics**

Only the multiplication relation and the Index structure matters

$\mathfrak{A} = \{a, b, c, d, \ldots\}$ **Alphabet**

$a < b < c < d < \ldots$ **ordered**

$\mathfrak{A}^*(\mathfrak{A})$ **Set of all words W**

$W = a_1 \cdot a_2 \cdot a_{27} \ldots a_{532} \equiv$ **concatenation product (nc)**

$W = p \cdot x \cdot s$ **p** $=$ **prefix; s** $=$ **suffix**

Definition:

A Lyndon word is smaller than any of its suffixes.

Theorem:[Radford, 1979]

The shuffle algebra $K\langle\mathfrak{A}\rangle$ is freely generated by the Lyndon words.
I.e. the number of Lyndon words yields the number of basic elements.

Examples :

$\{a, a, \ldots, a, b\} = aaa \ldots ab$ 1 Lyndon word for these sets

$n \quad a's: \quad n_{basic}/n_{all} = 1/n \qquad n \equiv$ depth of the sums

Symmetries lead to a smaller fraction.

Is there a general Counting Relation ?

E. Witt, 1937

$$l_n(n_1,\dots,n_q) = \frac{1}{n}\sum_{d|n_i}\mu(d)\frac{(n/d)!}{(n_1/d)!\dots(n_q/d)!}, \qquad \sum_i n_i = n$$

$\mu(k)$ Möbius function

2nd Witt formula.

The Length of the Basis is a function mainly of the Depth.

Observation: Sums with index -1 do not occur.

$$N_{\neg -1}(w) = \frac{1}{2}\left[\left(1+\sqrt{2}\right)^w + \left(1-\sqrt{2}\right)^w\right]$$

$$N_{\neg -1}^{\rm basic}(w) = \frac{2}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)N_{\neg -1}(d)$$

J.B., 2004; Further Reduction: Structural Relations.

Weight	Sums	a-basic	Sums $\neg -1$	a-basic	str. Rel.	Fraction
1	2	2	1	0	0	0.0
2	6	3	3	0	0	0.0
3	18	8	7	2	2	0.1111
4	54	18	17	5	3	0.0555
5	162	48	41	14	8	0.0494
6	486	116	99	28	?	<0.0576
	728	195	168	49	<41	<0.0563

The final set of functions:

Trivial functions:

$$S_{\pm k}(N) \quad \longrightarrow \quad \psi^{(k-1)}(N+1)$$

For $w = 1, 2$ no non–trivial functions contribute to the anomalous dimensions and Wilson coefficients.

Non–trivial functions:

$N = 3$: Two–Loop anomalous dimensions

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N)$$

Yndurain et al., 1980

$N = 4$: Two–Loop Wilson Coefficients

$$\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1-x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{S}_{1,2}(x)}{1\pm x}\right](N)$$

Structure Fct.: J.B., S. Moch, 2003,

Drell-Yan, Higgs-Prod., Fragmentation: J.B., V. Ravindran, 2004.

$N = 5$: Three–Loop Anomalous Dimensions

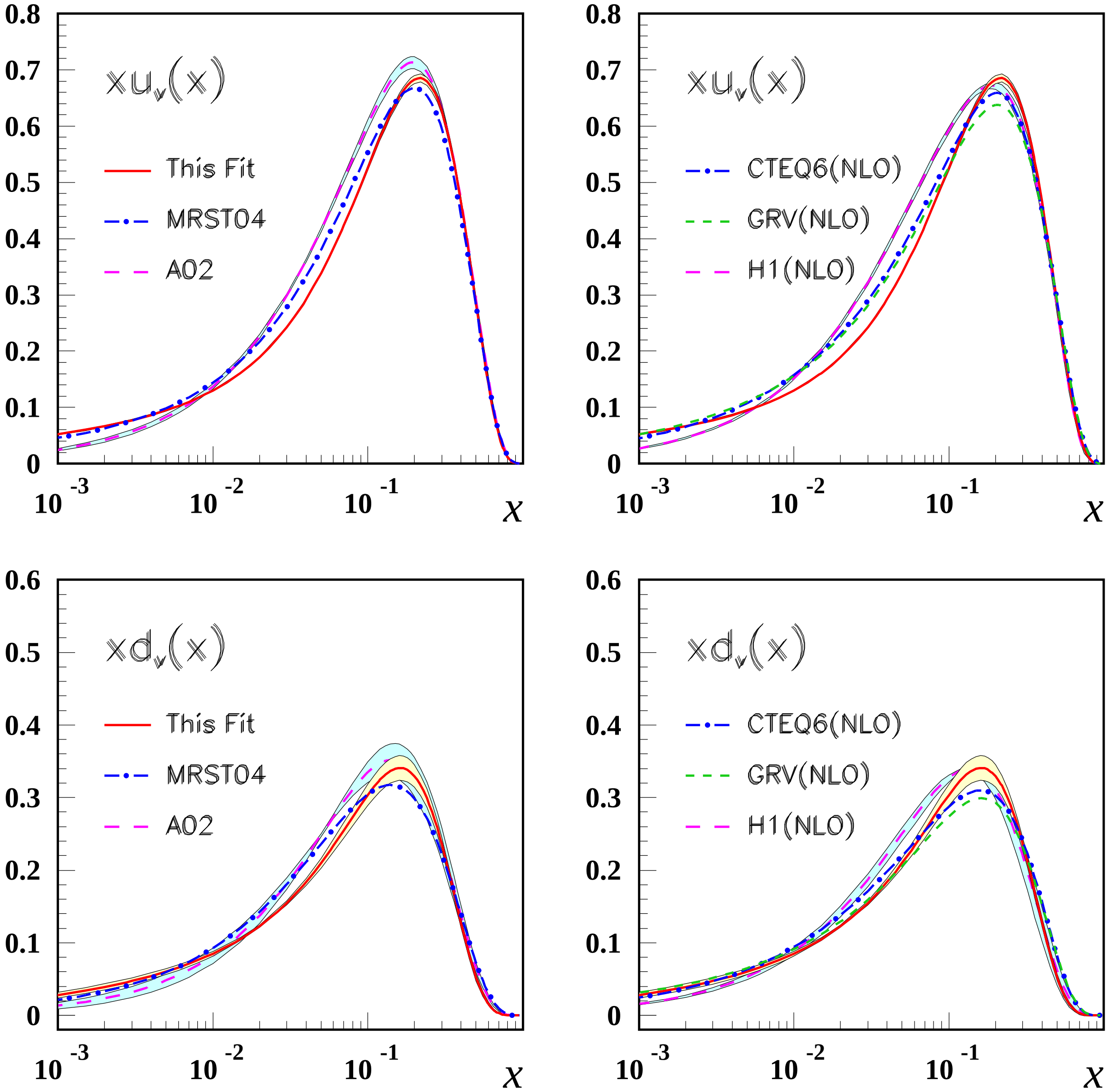
$$\mathbf{M}\left[\frac{\mathrm{Li}_4(x)}{1\pm x}\right](N), \quad \mathbf{M}\left[\frac{S_{1,3}(x)}{1+x}\right](N), \quad \mathbf{M}\left[\frac{S_{2,2}(x)}{1\pm x}\right](N),$$

$$\mathbf{M}\left[\frac{S_{2,2}(-x)-\mathrm{Li}_2^2(-x)/2}{1\pm x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{Li}_2^2(x)}{1+x}\right](N)$$

J.B., S. Moch, 2004.

Essentially 14 Functions seem to rule the single scale processes of massless QCD.

5. QCD NS-Analysis to 3 Loops



$W^2 > 12.5\,\text{GeV}^2, Q^2 > 4\,\text{GeV}^2$

NNLO :

$$\alpha_s(M_Z^2) = 0.1139^{+0.0026}_{-0.0028}$$

J.B., H. Böttcher, A. Guffanti, 2004

The World Data on F_2

$Experiment$	x	Q^2, GeV^2	F_2	$Norm$
BCDMS (100)	0.35 – 0.75	11.75 – 75.00	51	1.018
BCDMS (120)	0.35 – 0.75	13.25 – 75.00	59	1.011
BCDMS (200)	0.35 – 0.75	32.50 – 137.50	50	1.017
BCDMS (280)	0.35 – 0.75	43.00 – 230.00	49	1.018
NMC (comb)	0.35 – 0.50	7.00 – 65.00	15	1.003
SLAC (comb)	0.30 – 0.62	7.30 – 21.39	57	1.003
H1 (hQ2)	0.40 – 0.65	200 – 30000	26	1.018
ZEUS (hQ2)	0.40 – 0.65	650 – 30000	15	1.001
$proton$			322	
BCDMS (120)	0.35 – 0.75	13.25 – 99.00	59	0.992
BCDMS (200)	0.35 – 0.75	32.50 – 137.50	50	0.993
BCDMS (280)	0.35 – 0.75	43.00 – 230.00	49	0.993
NMC (comb)	0.35 – 0.50	7.00 – 65.00	15	0.980
SLAC (comb)	0.30 – 0.62	10.00 – 21.40	59	0.980
$deuteron$			232	
BCDMS (120)	0.070 – 0.275	8.75 – 43.00	36	1.000
BCDMS (200)	0.070 – 0.275	17.00 – 75.00	29	1.000
BCDMS (280)	0.100 – 0.275	32.50 – 115.50	27	1.000
NMC (comb)	0.013 – 0.275	4.50 – 65.00	88	1.000
SLAC (comb)	0.153 – 0.293	4.18 – 5.50	28	1.000
$non-singlet$			208	
$total$			762	

- **CUTS**: $0.3 < x < 1.0$ for F_2^p and F_2^d
  $0.0 < x < 0.3$ for $F_2^{ns} = 2(F_2^p - F_2^d)$
  $4.0 < Q^2 < 30000\, GeV^2$, $W^2 > 12.5\, GeV^2$

Fully Correlated Error Calculation

- The fully correlated 1σ error for the parton density f_q as given by Gaussian error propagation is

$$\sigma(f_q(x))^2 = \sum_{i,j=1}^{n_p} \left(\frac{\partial f_q}{\partial p_i} \frac{\partial f_q}{\partial p_j} \right) \mathrm{cov}(p_i, p_j) \ , \qquad (1)$$

  where the $\partial f_q / \partial p_i$ are the derivatives of f_q w.r.t. the parameters p_i and the $\mathrm{cov}(p_i, p_j)$ are the elements of the covariance matrix as determined in the fit.

- The derivatives $\partial f_q / \partial p_i$ at the input scale Q_0^2 can be calculated analytically. Their values at Q^2 are given by evolution.

- The derivatives evolved in MELLIN-N space are transformed back to x–space and can then be used according to the error propagation formula above.

$\Longrightarrow$ As an example the derivative of $f(x, a, b)$ w.r.t. parameter a in MELLIN–N space reads:

Fit Results

- Parameter values and Covariance Matrix at the input scale $Q_0^2 = 4.0\,GeV^2$

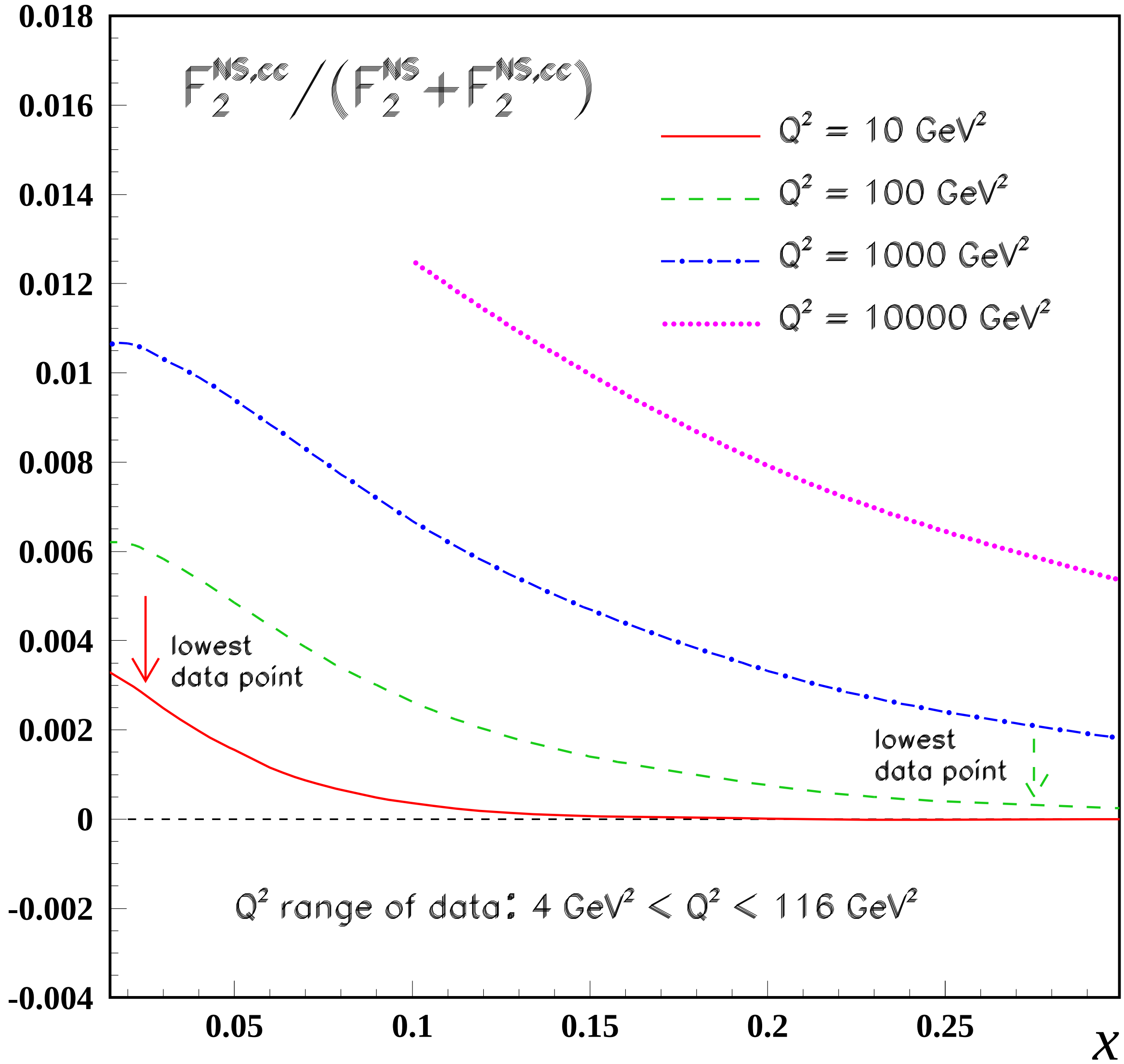
$$xq_i(x,Q_0^2) = A_i x^{a_i}(1-x)^{b_i}(1+\rho_i x^{\frac{1}{2}} + \gamma_i x)$$

u_v	a	0.299 $\pm$ 0.007
	b	4.157 $\pm$ 0.031
	ρ	0.751
	γ	28.833
d_v	a	0.488 $\pm$ 0.048
	b	6.609 $\pm$ 0.332
	ρ	-1.690
	γ	17.247
$\Lambda_{QCD}^{(4)}$		233 $\pm$ 34 MeV
χ^2/ndf		$=$ 630/757 $=$ 0.83

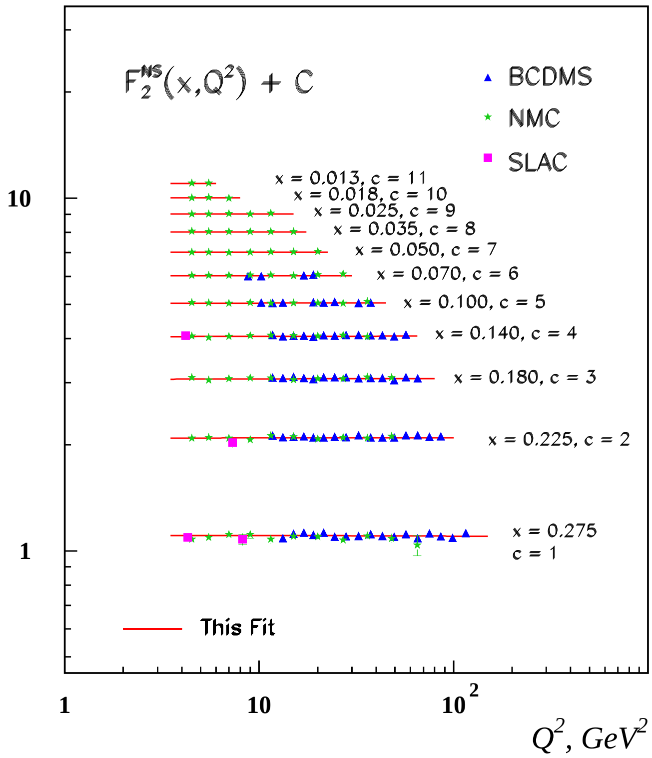
- Covariance Matrix at the input scale $Q_0^2 = 4.0\,GeV^2$

	$\Lambda_{QCD}^{(4)}$	a_{u_v}	b_{u_v}	a_{d_v}	b_{d_v}
$\Lambda_{QCD}^{(4)}$	**1.15E-3**				
a_{u_v}	1.03E-4	**5.40E-5**			
b_{u_v}	-8.45E-5	1.71E-4	**9.59E-4**		
a_{d_v}	4.17E-4	8.84E-6	-4.35E-4	**2.32E-3**	
b_{d_v}	2.32E-3	4.21E-4	-2.28E-3	1.48E-2	**1.10E-1**

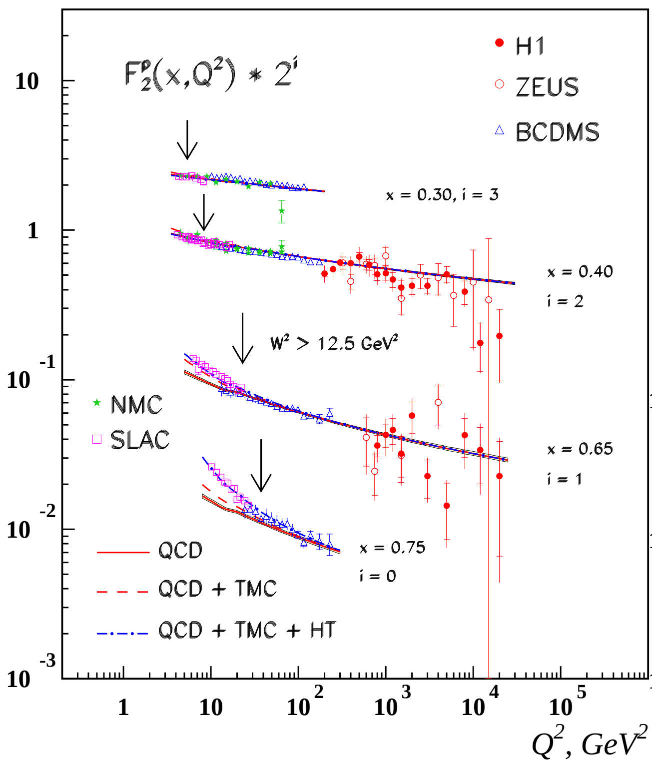
Heavy Flavor NS-contributions



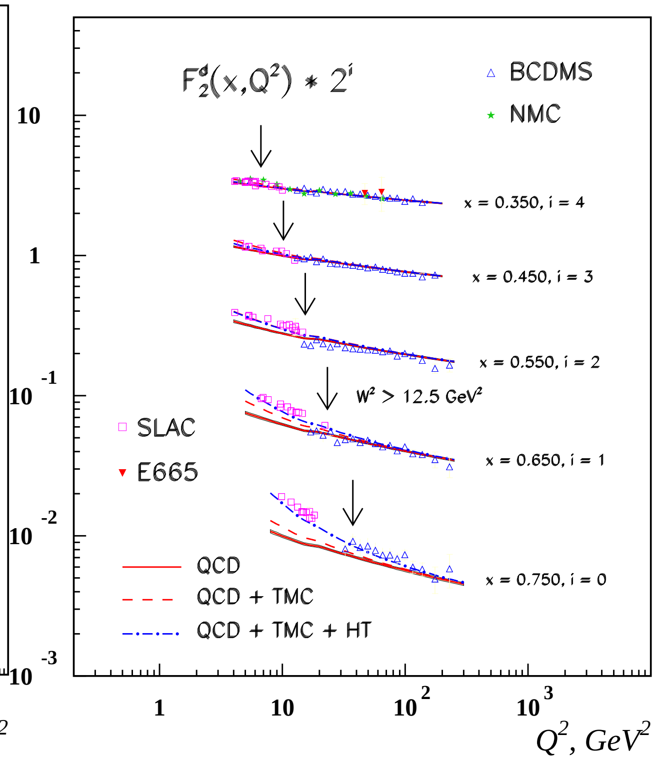
NON-SINGLET 3-LOOP QCD ANALYSIS



$x > 0.3$



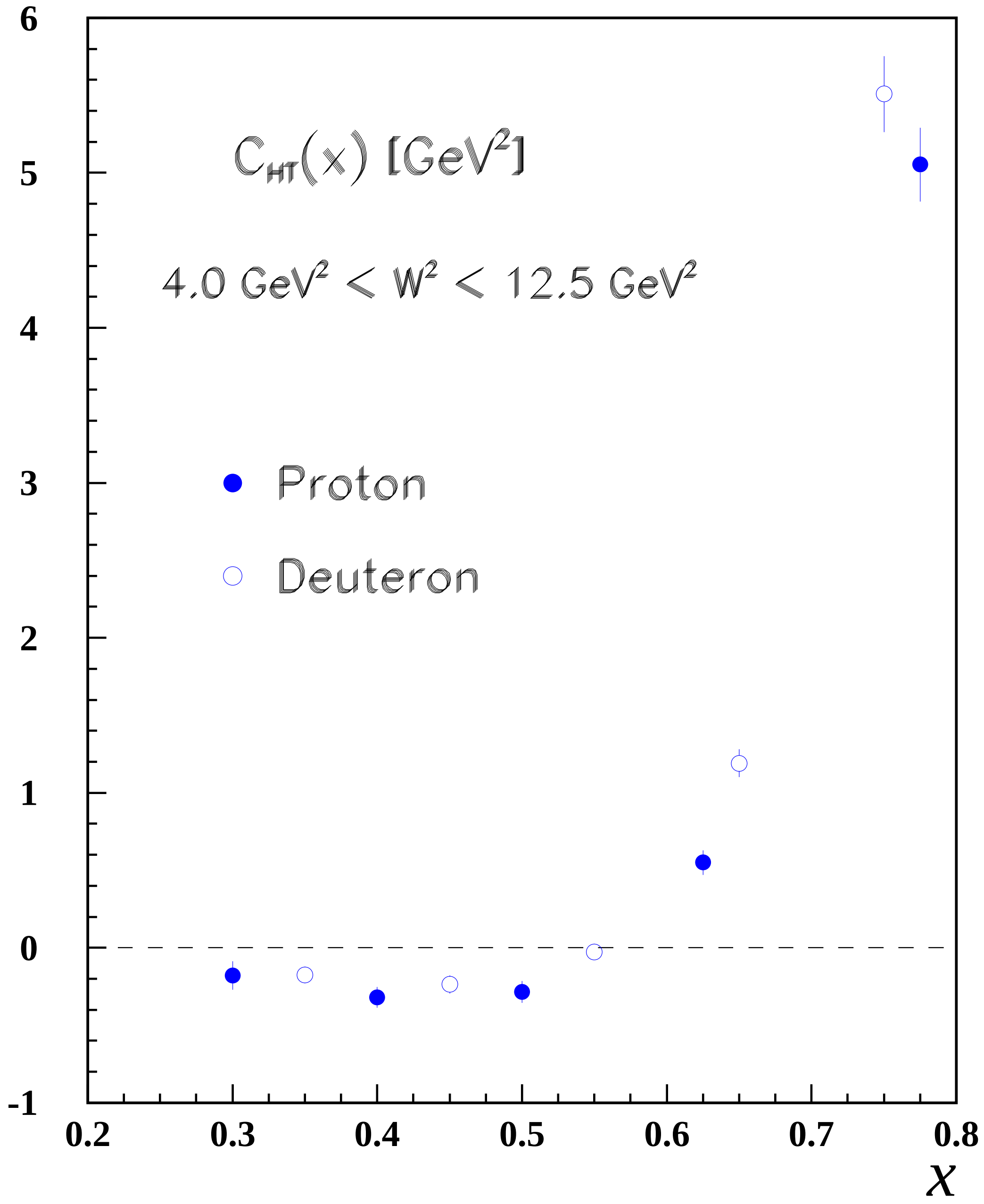
proton



deuteron

Higher Twist Contributions:

$$4 < W^2 < 12.5\,\text{GeV}^2, Q^2 > 4\,\text{GeV}^2$$



Moments and Lattice Results

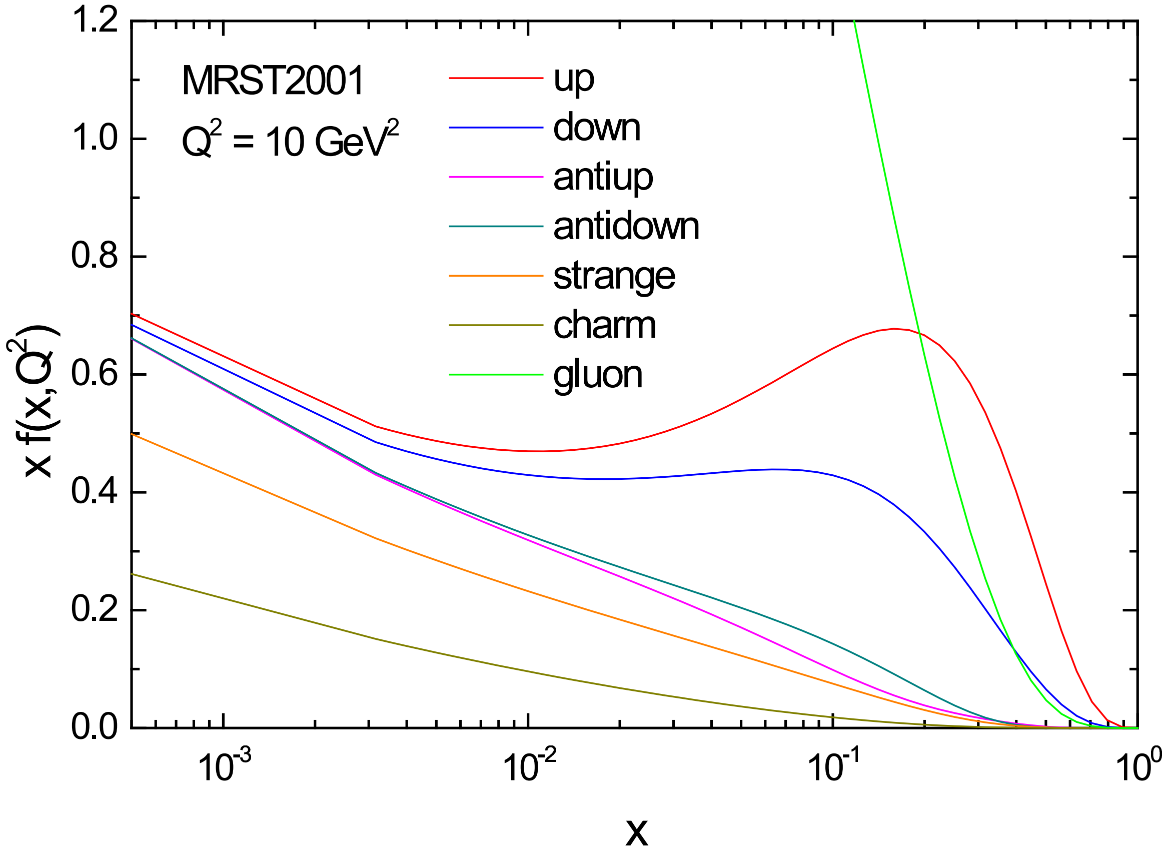
f	n	This Fit	MRST04	A02
u_v	2	0.288 ± 0.003	0.285	0.304
	3	0.084 ± 0.001	0.082	0.087
	4	0.0319 ± 0.0004	0.032	0.033
d_v	2	0.113 ± 0.004	0.115	0.120
	3	0.026 ± 0.001	0.028	0.028
	4	0.0078 ± 0.0004	0.009	0.010
$u_v - d_v$	2	0.175 ± 0.004	0.171	0.184
	3	0.058 ± 0.001	0.055	0.059
	4	0.0241 ± 0.0005	0.022	0.024

First lattice results on $u_v - d_v, N = 2$ yield promising values using overlap-fermions (QCDSF).

More results also are upcoming.

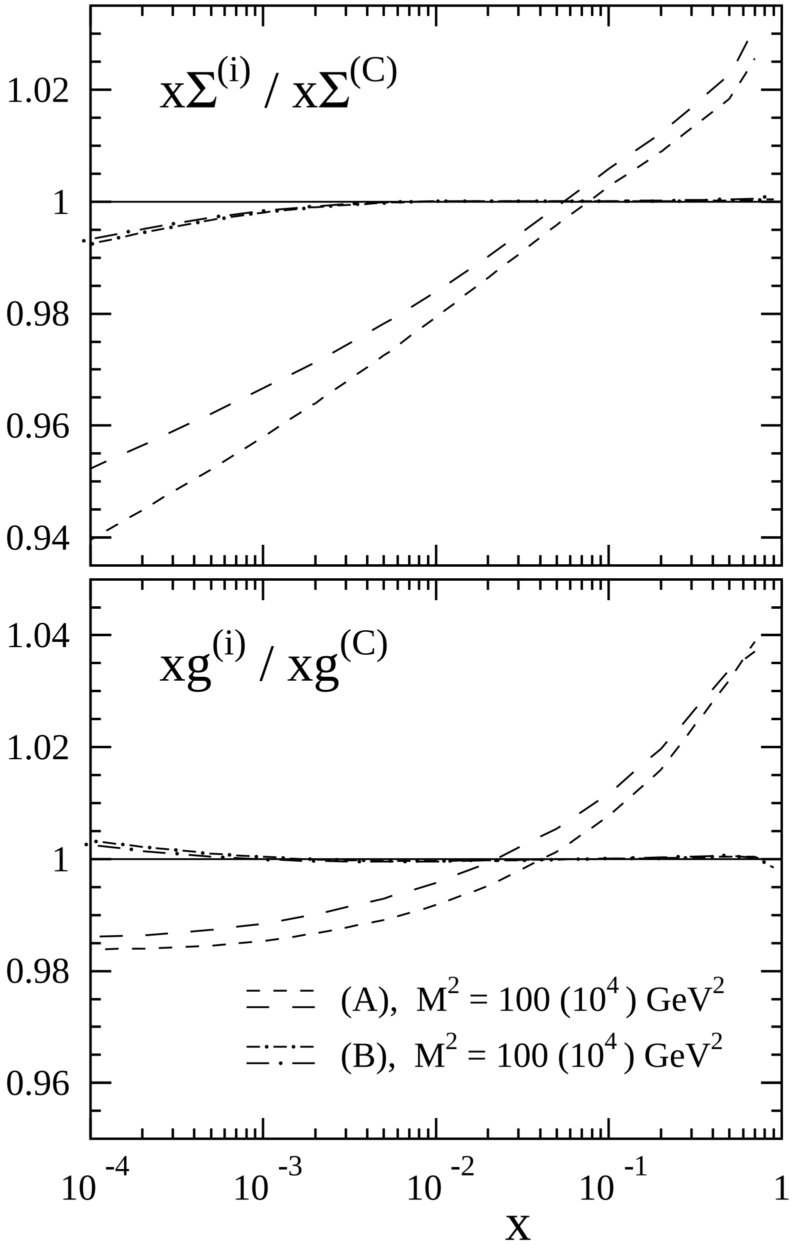
6. The Singlet Sector

Parton Densities: Relative Size



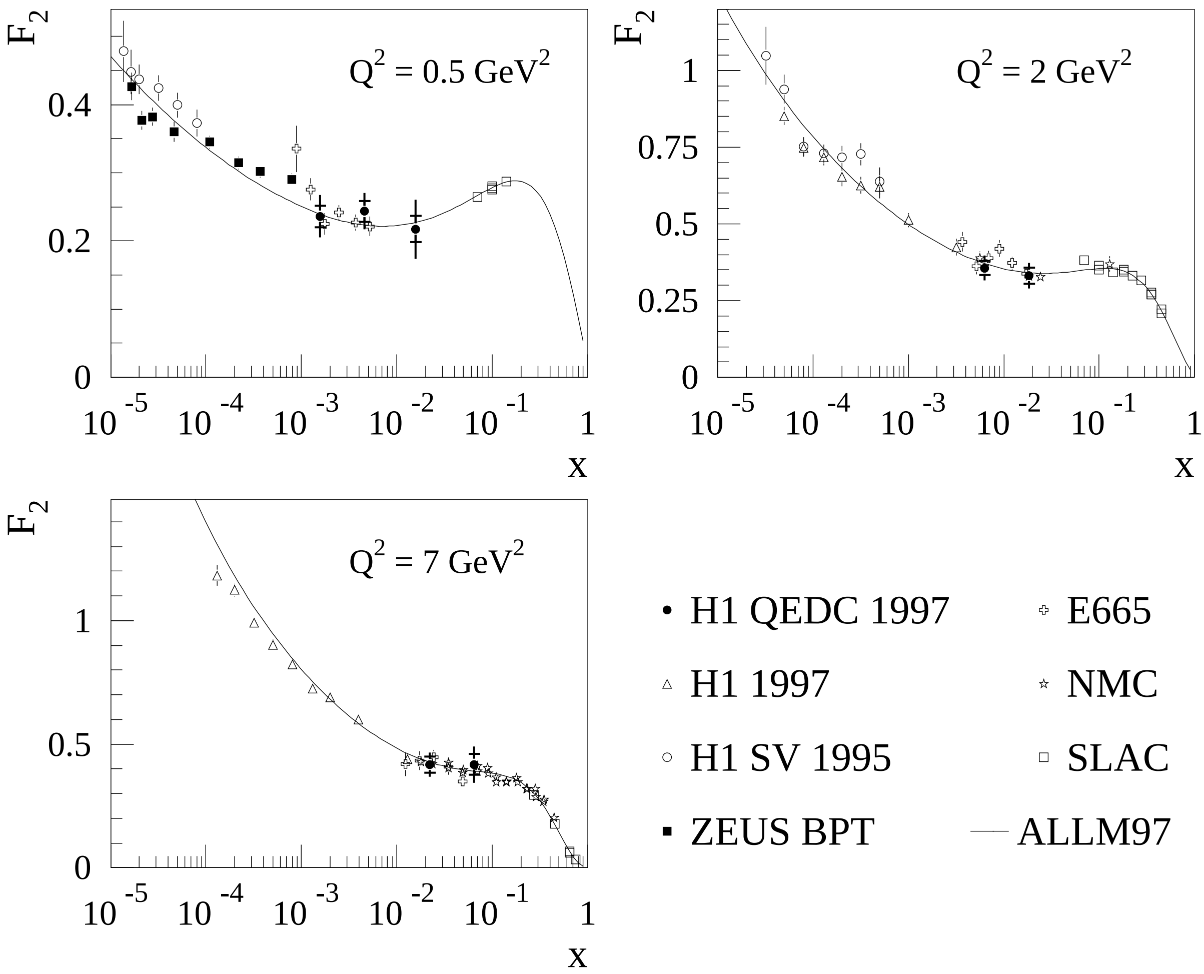
Pile–Up Effects:

Iterative vs Exact Solution of Evolution Equations

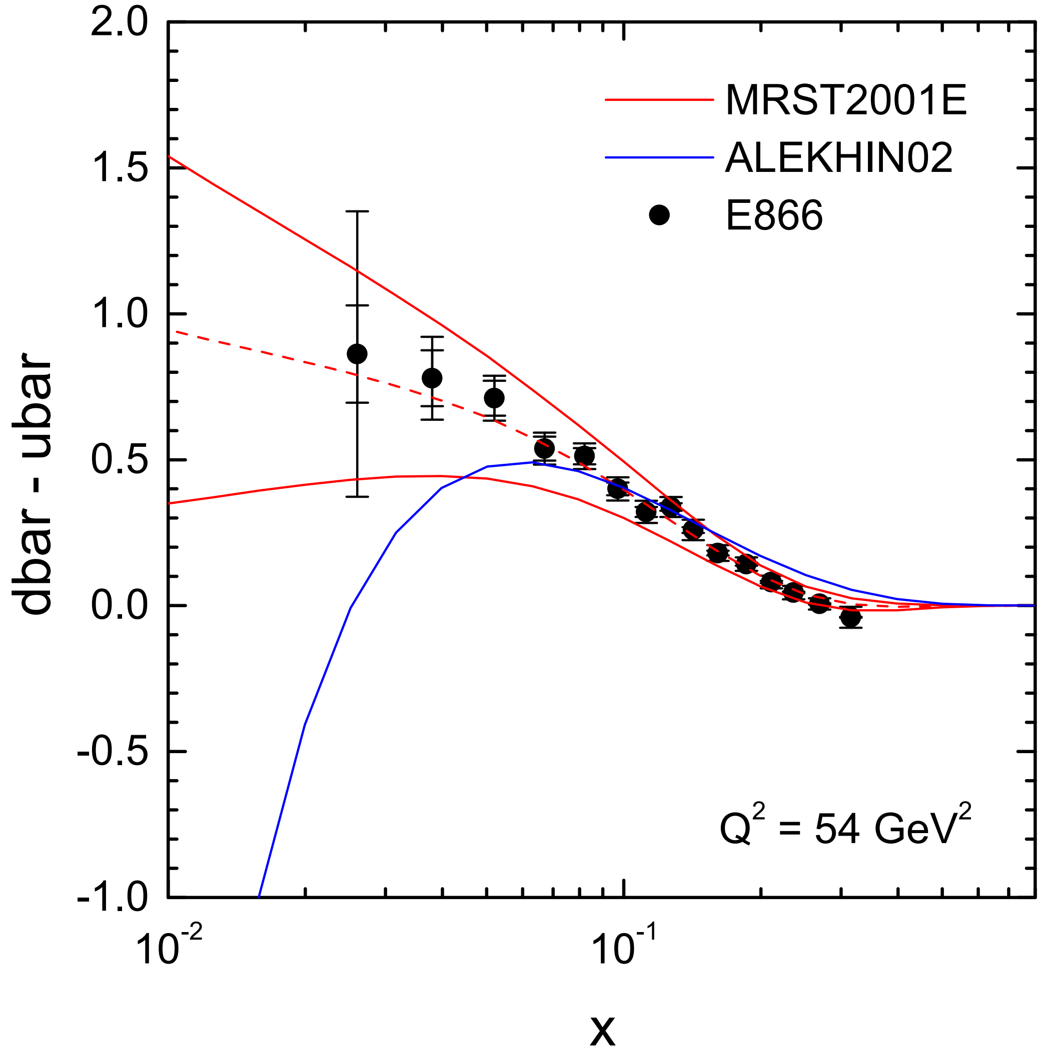


Blümlein, Riemersma, van Neerven, Vogt, 1996

x rise of $F_2(x, Q^2)$ at low Q^2 :

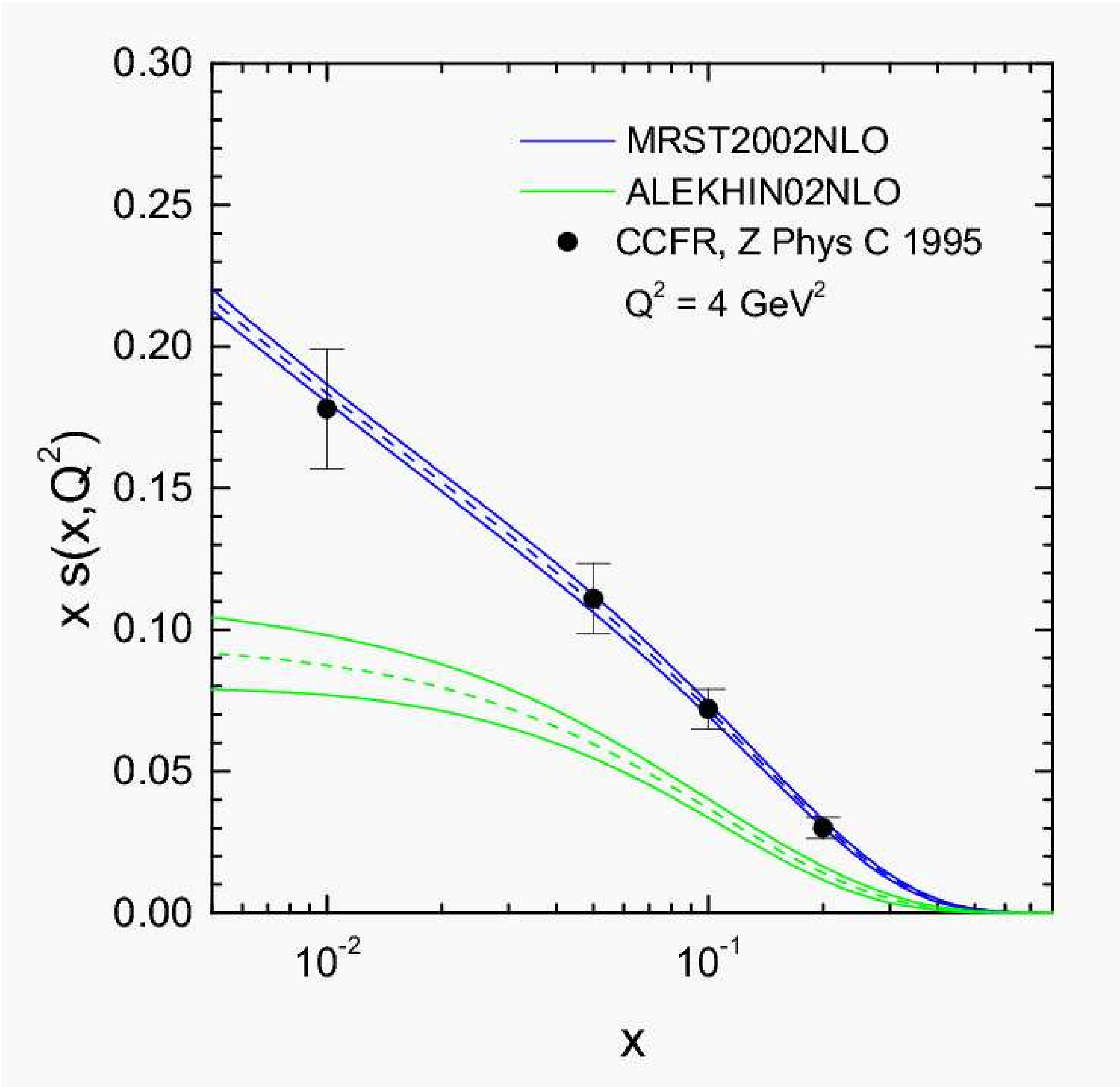


$\overline{d} - \overline{u}$



$x(\overline{d}(x) - \overline{u}(x)) = 1.195x^{1.24}(1-x)^{9.10}(1 + 14.05x - 45.52x^2)$
$Q^2 = 1$ GeV^2

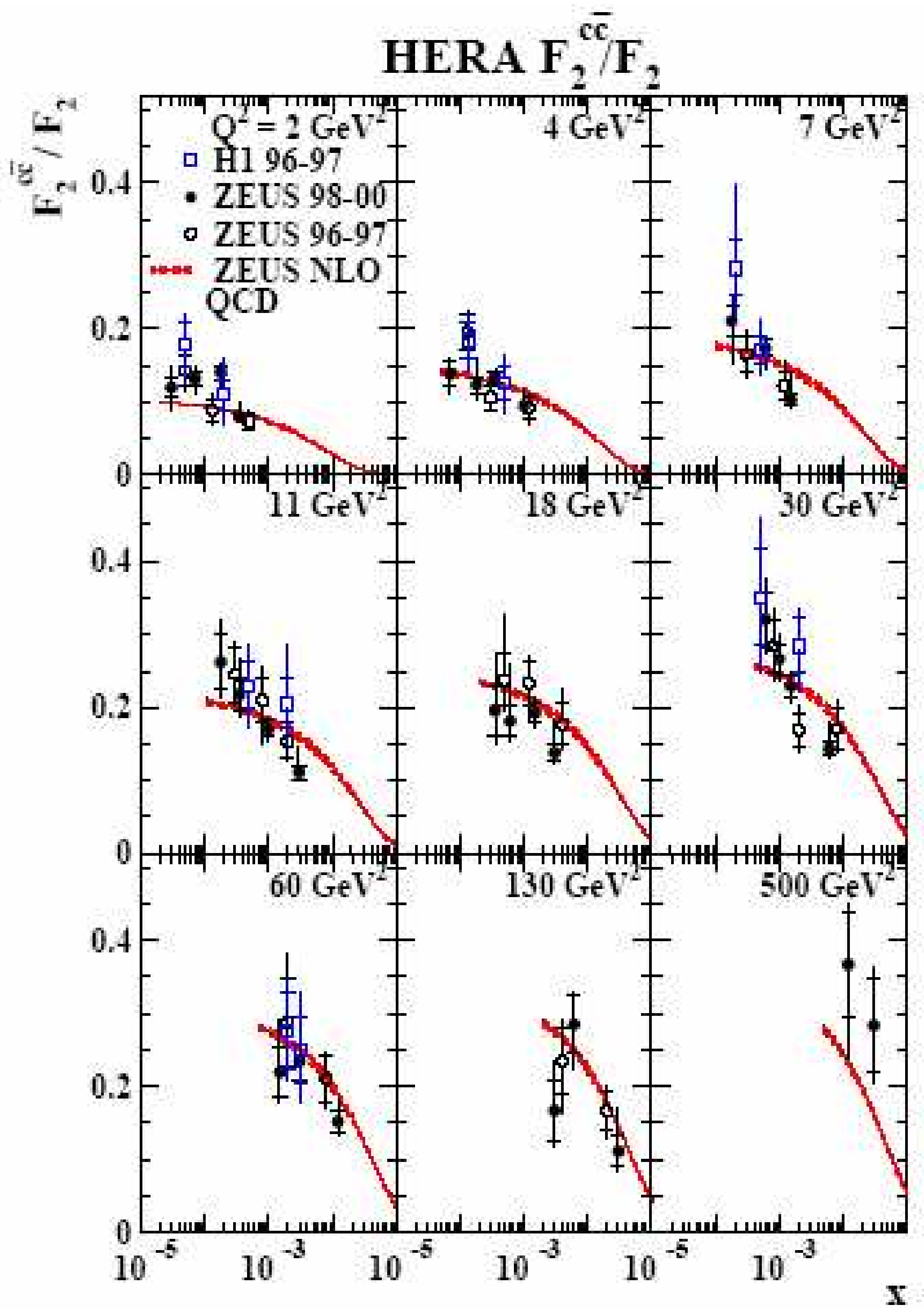
Strange quark distribution



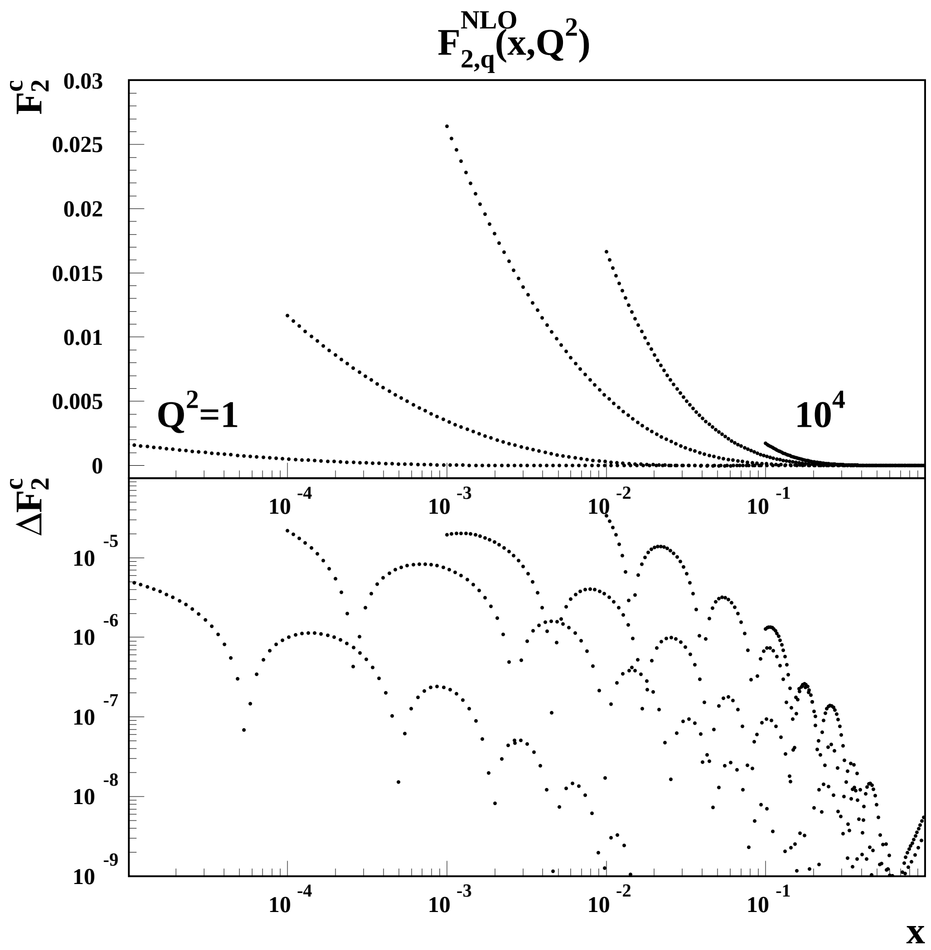
- CCFR : iron target, EMC effect. How large ?

CAN HERMES MEASURE $s(x, Q^2)$?

$c\bar{c}$ Structure Function F_2



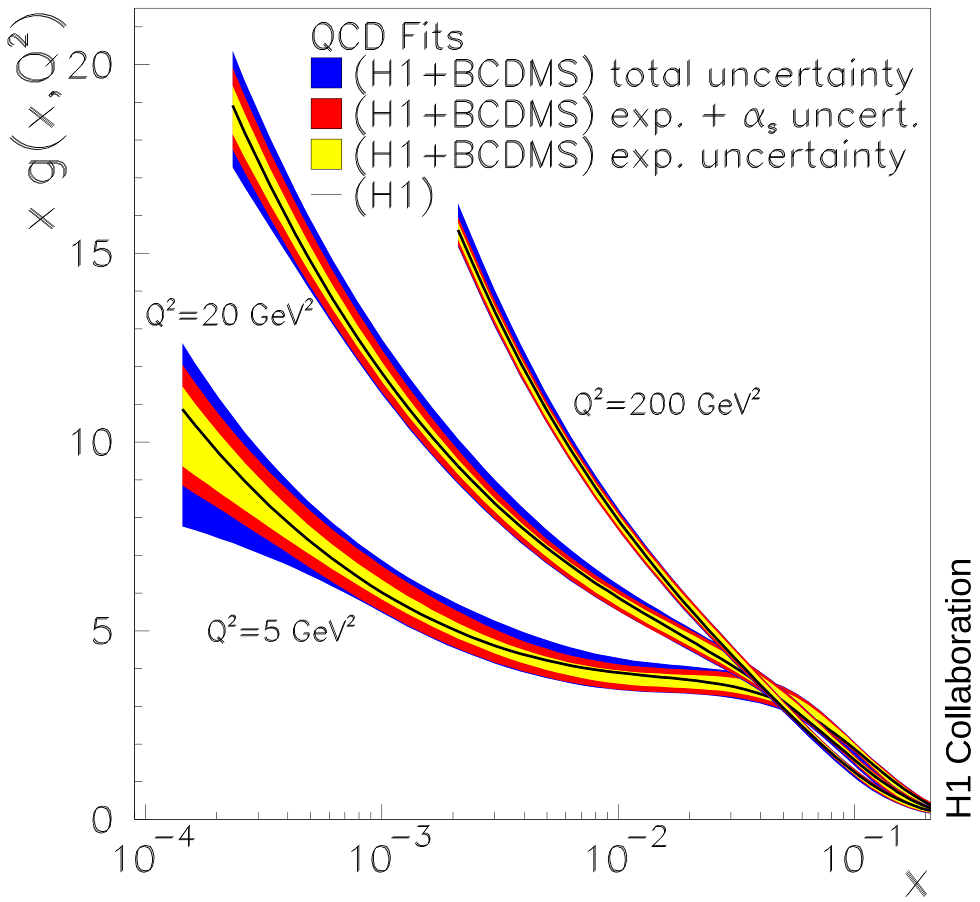
Mellin-space representation :

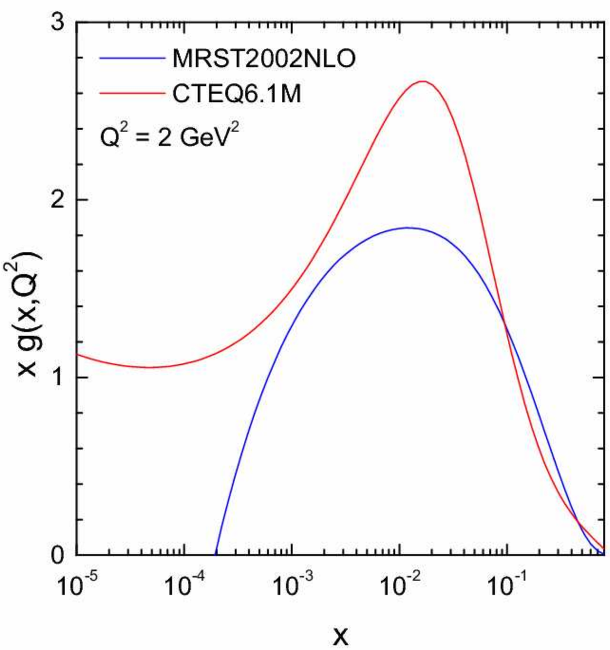


S. Alekhin and J.B., 2004

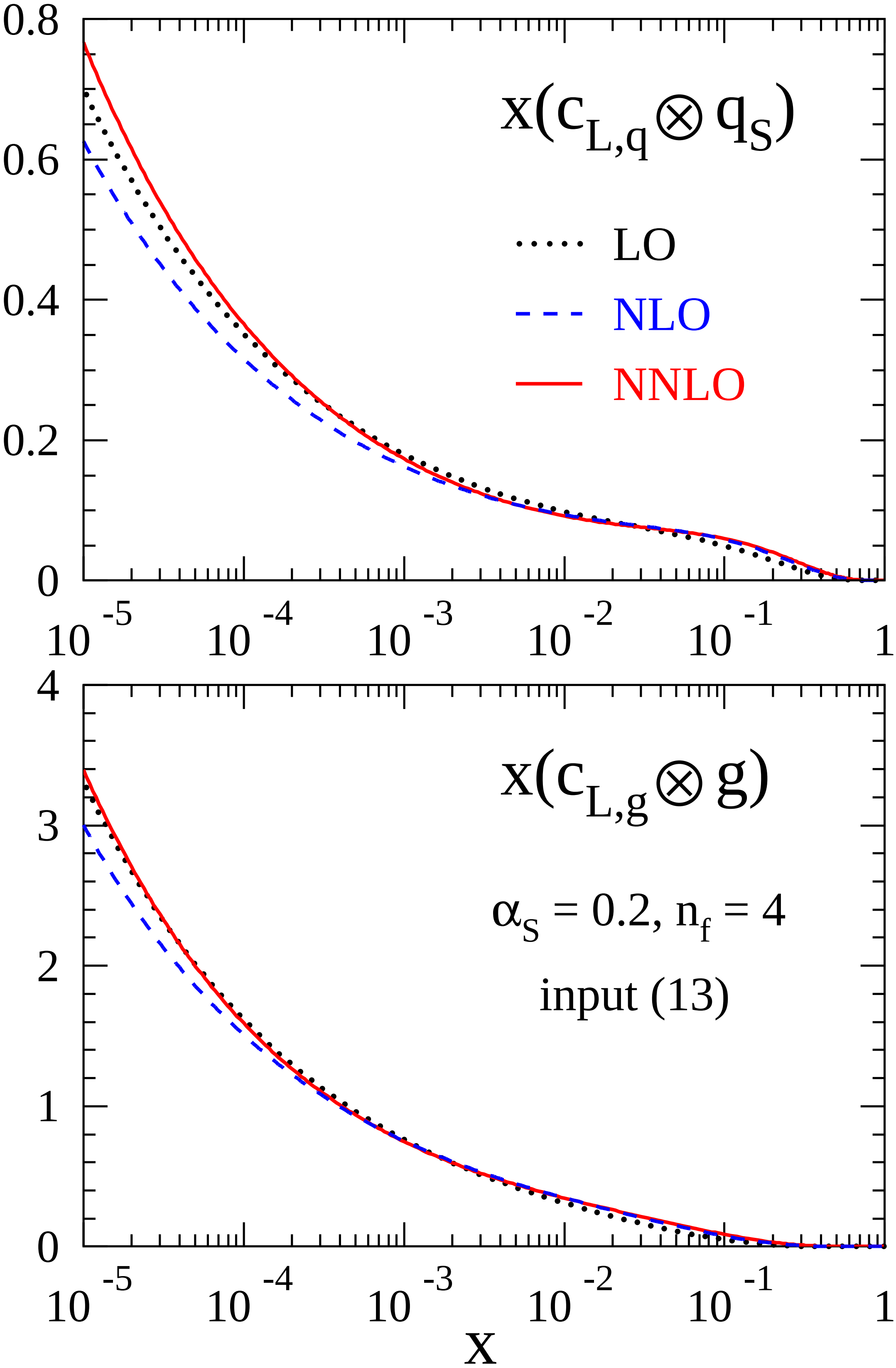
- necessary for scheme-invariant evolution.
- fast and accurate access to heavy flavor Wilson coefficients.

Gluon Density



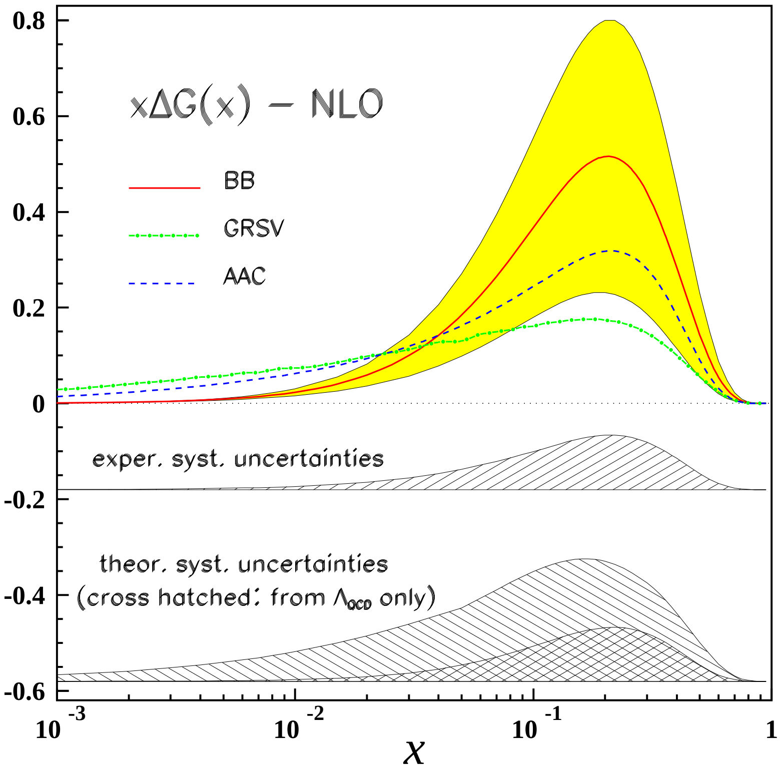


$F_L(x, Q^2)$

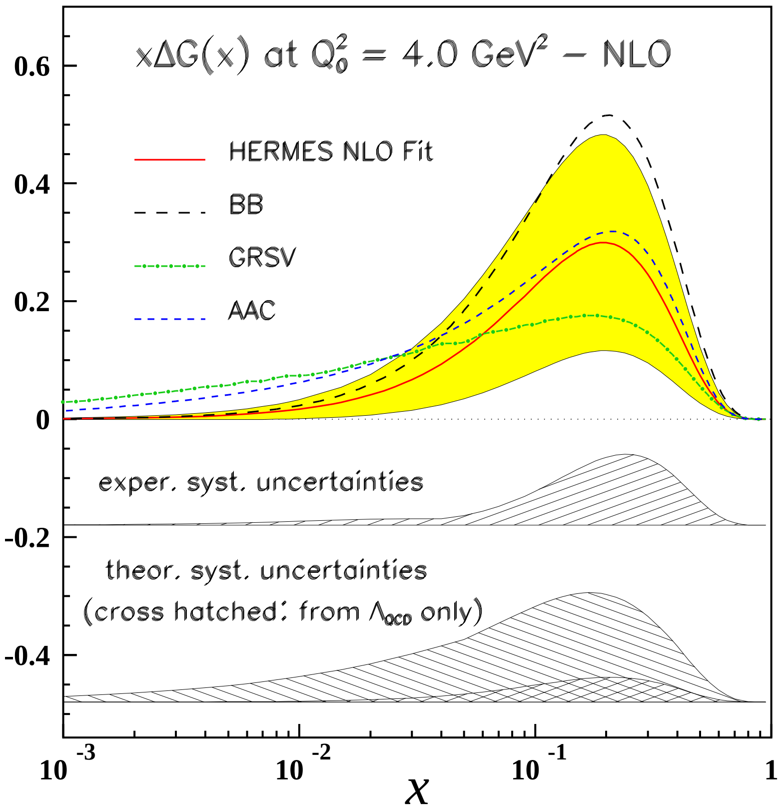


Moch, Vermasern, Vogt, hep-ph/0411112

Gluon Distribution:HERMES



$\alpha_s(M_Z^2) = 0.1135$



$\alpha_s(M_Z^2) = 0.1182$

Scheme–invariant Evolution Equations

Evolution Equations of Structure or Fragmentation Functions do normally exhibit FACTORIZATION and RENORMALIZATION SCHEME dependences. Instead of PROCESS-INDEPENDENT SCHEME-DEPENDENT Evolution equations for PARTONS one may think of PROCESS-DEPENDENT SCHEME-INDEPENDENT EVOLUTION EQUATIONS FOR **Observables.**

Evolution Equations :

$$\frac{\partial}{\partial t}\begin{pmatrix} F_A^N \\ F_B^N \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} K_{AA}^N & K_{AB}^N \\ K_{BA}^N & K_{BB}^N \end{pmatrix}\begin{pmatrix} F_A^N \\ F_B^N \end{pmatrix},$$

evolution variable

$$t = -\frac{2}{\beta_0}\ln\left(\frac{a_s(Q^2)}{a_s(Q_0^2)}\right),$$

physical evolution kernels

$$K_{IJ}^N = \left[-4\frac{\partial C_{I,m}^N(t)}{\partial t}\left(C^N\right)^{-1}_{m,J}(t) - \frac{\beta_0 a_s(Q^2)}{\beta(a_s(Q^2))}C_{I,m}^N(t)\gamma_{mn}^N(t)\left(C^N\right)^{-1}_{n,J}(t)\right]$$

with

$$K_{IJ}^N = \sum_{n=0}^{\infty} a_s^n(Q^2)\left(K^N\right)^{(n)}_{IJ}$$

Possible choices for F_A and F_B are F_2 and $\partial F_2/\partial t$ or F_2 and F_L. For these sets of physical observables we will examine the crossing-behaviour from S to T-Channel.

The dependence on the **renormalization scheme** is only removed if the perturbation series is summed to all orders.

System : $F_2(x,Q^2), \partial F_2/\partial t(x,Q^2)$

Leading Order :

$$
\begin{aligned}
K_{22}^{N(0)} &= 0 \\
K_{2d}^{N(0)} &= -4 \\
K_{d2}^{N(0)} &= \frac{1}{4}\left(\gamma_{qq}^{N(0)}\gamma_{gg}^{N(0)} - \gamma_{qg}^{N(0)}\gamma_{gq}^{N(0)}\right) \\
K_{dd}^{N(0)} &= \gamma_{qq}^{N(0)} + \gamma_{gg}^{N(0)}
\end{aligned}
$$

Next-to-Leading Order :

[Furmanski, Petronzio 1982]

$$
\begin{aligned}
K_{22}^{N(1)} &= K_{2d}^{N(1)} = 0 \\
K_{d2}^{N(1)} &= \frac{1}{4}\left[\gamma_{gg}^{N(0)}\gamma_{qq}^{N(1)} + \gamma_{gg}^{N(1)}\gamma_{qq}^{N(0)} - \gamma_{qg}^{N(1)}\gamma_{gq}^{N(0)} - \gamma_{qg}^{N(0)}\gamma_{gq}^{N(1)}\right] \\
&\quad - \frac{\beta_1}{2\beta_0}\left(\gamma_{qq}^{N(0)}\gamma_{gg}^{N(0)} - \gamma_{gq}^{N(0)}\gamma_{qg}^{N(0)}\right) \\
&\quad + \frac{\beta_0}{2}C_{2,q}^{N(1)}\left(\gamma_{qq}^{N(0)} + \gamma_{gg}^{N(0)} - 2\beta_0\right) \\
&\quad - \frac{\beta_0}{2}\frac{C_{2,g}^{N(1)}}{\gamma_{qg}^{N(0)}}\left[(\gamma_{qq}^{N(0)})^2 - \gamma_{qq}^{N(0)}\gamma_{gg}^{N(0)} + 2\gamma_{qg}^{N(0)}\gamma_{gq}^{N(0)} - 2\beta_0\gamma_{qq}^{N(0)}\right] \\
&\quad - \frac{\beta_0}{2}\left(\gamma_{qq}^{N(1)} - \frac{\gamma_{qq}^{N(0)}\gamma_{qg}^{N(1)}}{\gamma_{qg}^{N(0)}}\right)
\end{aligned}
\tag{1}
$$

$$K_{dd}^{N(1)} = \gamma_{qq}^{N(1)} + \gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0}\left(\gamma_{qq}^{N(0)} + \gamma_{gg}^{N(0)}\right) + 4\beta_0 C_{2,q}^{N(1)} - 2\beta_1$$
$$- \frac{2\beta_0}{\gamma_{qg}^{N(0)}}\left[C_{2,g}^{N(1)}\left(\gamma_{qq}^{N(0)} - \gamma_{gg}^{N(0)} - 2\beta_0\right) - \gamma_{qg}^{N(1)}\right]$$

System : $F_2(x, Q^2), F_L(x, Q^2)$

$(\widetilde{F}_L^N \equiv F_L^N/(a_s(Q^2)C_{L,g}^{N(1)}))$

Leading Order :

[Catani 1997]

$$K_{22}^{N(0)} = \gamma_{qq}^{N(0)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\gamma_{qg}^{N(0)}$$
$$K_{2L}^{N(0)} = \gamma_{qg}^{N(0)}$$
$$K_{L2}^{N(0)} = \gamma_{gq}^{N(0)} - \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^2 \gamma_{qg}^{N(0)}$$
$$K_{LL}^{N(0)} = \gamma_{gg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\gamma_{qg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\left(\gamma_{qq}^{N(0)} - \gamma_{gg}^{N(0)}\right)$$

Next-to-Leading Order :

[BRvN 2000]

$$K_{22}^{N(1)} = \gamma_{qq}^{N(1)} - \frac{\beta_1}{\beta_0}\gamma_{qq}^{N(0)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\left(\gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0}\gamma_{qg}^{N(0)}\right)$$
$$+ \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)}\gamma_{qq}^{N(0)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)}\gamma_{gg}^{N(0)}$$

$$
\begin{aligned}
& - \left[\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^2 C_{2,g}^{N(1)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}}\right] \gamma_{qg}^{N(0)} \\
& + C_{2,g}^{N(1)} \gamma_{gq}^{N(0)} + 2\beta_0 \left(C_{2,q}^{N(1)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)}\right) \\
K_{2L}^{N(1)} = & \ \gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qg}^{N(0)} - C_{2,g}^{N(1)} (\gamma_{qq}^{N(0)} - \gamma_{gg}^{N(0)}) + 2\beta_0 C_{2,g}^{N(1)} \\
& + \left(C_{2,q}^{N(1)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} - \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}}\right) \gamma_{qg}^{N(0)} \\
K_{L2}^{N(1)} = & \ \gamma_{gq}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{gq}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \left(\gamma_{qq}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qq}^{N(0)}\right) \\
& - \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^2 \left(\gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qg}^{N(0)}\right) \\
& - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \left(\gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{gg}^{N(0)}\right) \\
& + \left[\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,q}^{N(1)} + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^2 C_{2,g}^{N(1)}\right] \gamma_{qq}^{N(0)} \\
& - \left[\left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^3 C_{2,g}^{N(1)} + 2 \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^2 \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}}\right. \\
& \left. - \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^2 C_{2,q}^{N(1)}\right] \gamma_{qg}^{N(0)} \\
& + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} - C_{2,q}^{N(1)} + \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}}\right) \gamma_{gq}^{N(0)}
\end{aligned}
$$

$$
\begin{aligned}
K_{LL}^{N(1)} \quad = \quad & \gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0}\gamma_{gg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\left(\gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0}\gamma_{qg}^{N(0)}\right) \\
& - \left[\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^2 C_{2,g}^{N(1)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,q}^{N(1)}\right] \gamma_{gg}^{N(0)} \\
& + 2\beta_0 \left(\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}}\right) \\
& - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \gamma_{qq}^{N(0)} + \left[\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}}\right. \\
& \left. + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^2 C_{2,g}^{N(1)}\right] \gamma_{qg}^{N(0)} \\
& - C_{2,g}^{N(1)} \gamma_{gq}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \gamma_{gg}^{N(0)} + 2\beta_0 \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}}
\end{aligned}
$$

3 Loop : (including heavy flavor) J.B. and A. Guffanti

Only one fit parameter. Input distributions measured.

7. Polarized Nucleons

HOW IS THE NUCLEON SPIN DISTRIBUTED OVER THE PARTONS?

$$S_n = \frac{1}{2}\left[\Delta(u+\bar{u}) + \Delta(d+\bar{d}) + \Delta(s+\bar{s})\right] + \Delta G + L_q + L_g$$

$$S_n = \frac{1}{2}$$

$$\begin{aligned} \Delta\Sigma &= 0.138 \pm 0.082, \quad (0.150 \pm 0.061) \\ \Delta G &= 1.026 \pm 0.554, \quad (0.931 \pm 0.679) \end{aligned}$$

EMC, 1987: THE NUCLEON SPIN IS NOT THE SUM OF THE LIGHT QUARK SPINS.

MEASURE:

POLARIZED PARTON DENSITIES: $\Delta q_i, \Delta G$

HOW CAN ONE ACCESS THE PARTON ANGULAR MOMENTUM ?

POLARIZED HEAVY FLAVOR CONTRIBUTIONS.

- POLARIZED STRUCTURE FUNCTIONS CONTAIN ALSO TWIST 3 CONTRIBUTIONS.

HOW TO UNFOLD THESE TERMS ?

Polarized Parton Densities:

pioneering work: Dortmund GRSV, 1996, 2001

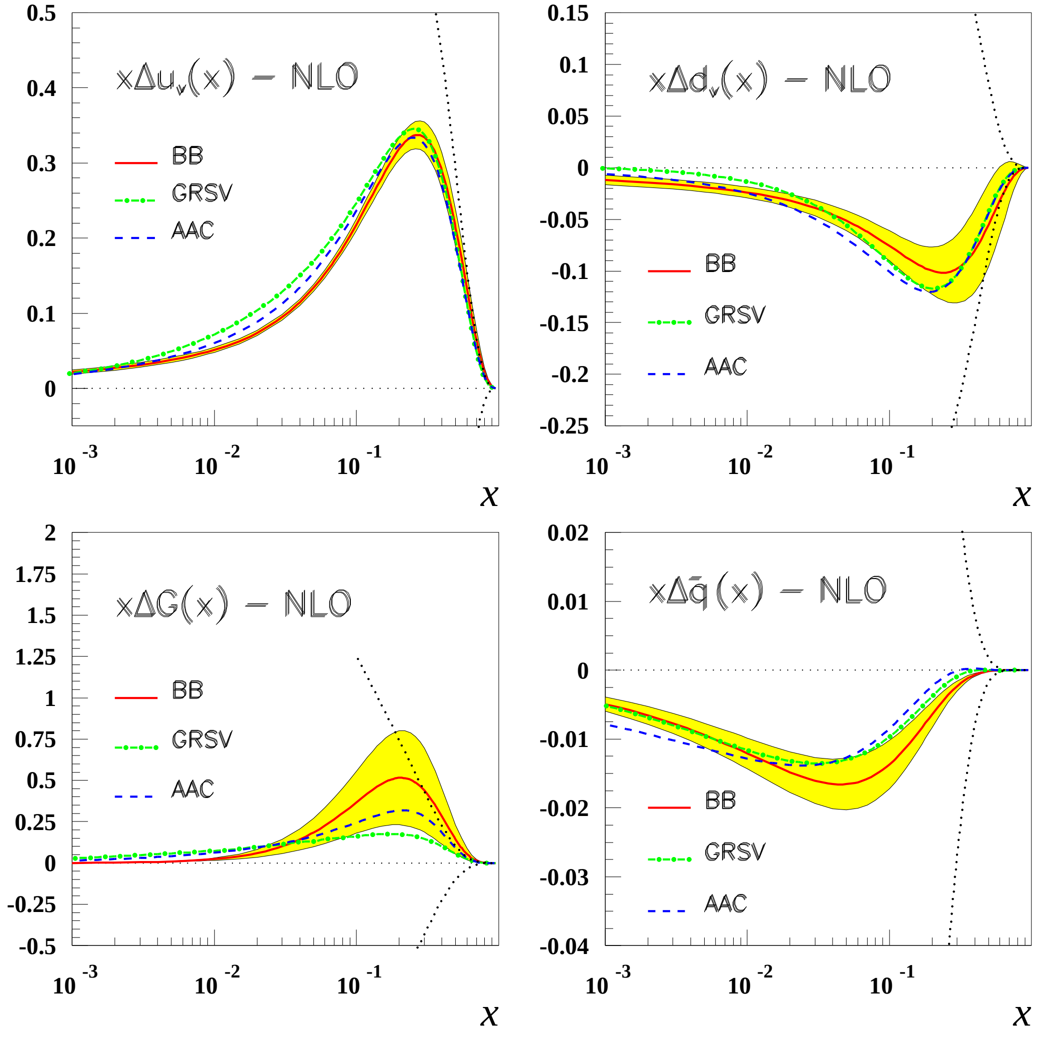
Analysis by other groups:

AAC (Japan), 2000, 2004

J.B., H. Böttcher, 2002

Leader et al., 2002

Altarelli et al., 1997



$$\text{NLO}: \quad \alpha_s(M_z^2) = 0.113^{+0.10}_{-0.08}$$

J.B., H. Böttcher, 2002

Comparison with Lattice Moments:

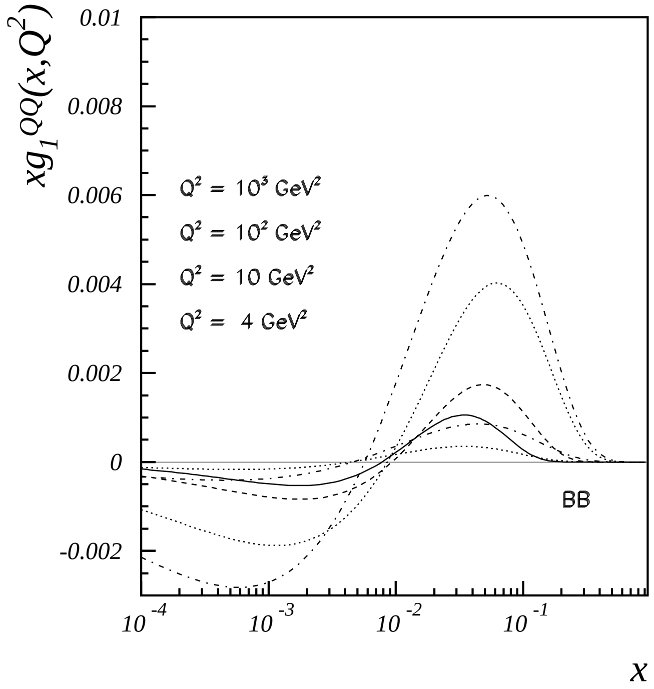
	Moment	BB, NLO	QCDSF	LHPC/SESAM
Δu_v	0	0.926	0.889 ± 0.029	0.860 ± 0.069
	1	0.163 ± 0.014	0.198 ± 0.008	0.242 ± 0.022
	2	0.055 ± 0.006	0.041 ± 0.009	0.116 ± 0.042
Δd_v	0	-0.341	-0.236 ± 0.027	-0.171 ± 0.043
	1	-0.047 ± 0.021	-0.048 ± 0.003	-0.029 ± 0.013
	2	-0.015 ± 0.009	-0.028 ± 0.002	0.001 ± 0.025
$\Delta u_v - \Delta d_v$	0	1.267	1.14 ± 0.03	1.031 ± 0.081
	1	0.210 ± 0.025	0.245 ± 0.009	0.271 ± 0.025
	2	0.070 ± 0.011	0.069 ± 0.009	0.115 ± 0.049

1st moments: Still problematic.

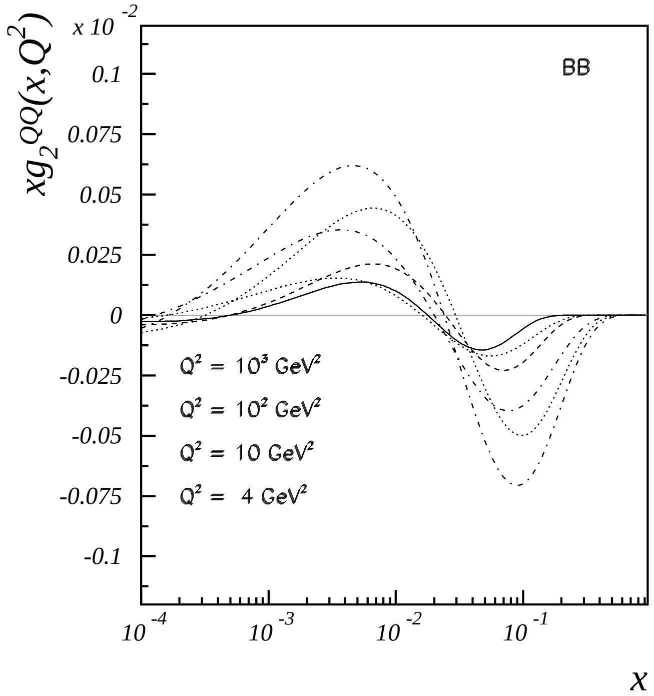
HEAVY FLAVOR:

g_1 : Watson, 1982; Vogelsang, 1990

g_2 : J.B., Ravindran, van Neerven, 2003



$g_1^{c\bar{c}}$



$g_2^{c\bar{c}}$

Sum Rules and Integral Relations:

Twist 2:

$$g_2(x,Q^2) = -g_1(x,Q^2) + \int_x^1 \frac{dy}{y} g_1(y,Q^2)$$

Wandzura, Wilczek, 1977;

Piccione, Ridolfi 1998; J.B., A. Tkabladze, 1998 : with TM

$$g_3(x,Q^2) = 2x \int_x^1 \frac{dy}{y^2} g_4(y,Q^2)$$

J.B., N. Kochelev, 1996; J.B., A. Tkabladze, 1998 : with TM

Twist 3:

Include nucleon mass effects.

J.B., A. Tkabladze, 1998

$$\begin{aligned}
g_1(x,Q^2) &= \frac{4M^2x^2}{Q^2}\left[g_2(x,Q^2) - 2\int_x^1 \frac{dy}{y} g_2(y,Q^2)\right] \\
\frac{4M^2x^2}{Q^2} g_3(x,Q^2) &= g_4(x,Q^2)\left(1 + \frac{4M^2x^2}{Q^2}\right) + 3\int_x^1 \frac{dy}{y} g_4(y,Q^2) \\
2x g_5(x,Q^2) &= -\int_x^1 \frac{dy}{y} g_4(y,Q^2)
\end{aligned}$$

8. $\Lambda_{\rm QCD}$ and $\alpha_s(M_Z^2)$

NLO	$\alpha_s(M_Z^2)$	expt	theory	Ref.
CTEQ6	0.1165	± 0.0065		[1]
MRST03	0.1165	± 0.0020	± 0.0030	[2]
A02	0.1171	± 0.0015	± 0.0033	[3]
ZEUS	0.1166	± 0.0049		[4]
H1	0.1150	± 0.0017	± 0.0050	[5]
BCDMS	0.110	± 0.006		[6]
BB (pol)	0.113	± 0.004	$^{+0.009}_{-0.006}$	[7]

NNLO	$\alpha_s(M_Z^2)$	expt	theory	Ref.
MRST03	0.1153	± 0.0020	± 0.0030	[2]
A02	0.1143	± 0.0014	± 0.0009	[3]
SY01(ep)	0.1166	± 0.0013		[8]
SY01(νN)	0.1153	± 0.0063		[8]
BBG	0.1139	$+0.0026/-0.0028$		[9]

BBG: $N_f = 4$: non-singlet data-analysis at $O(\alpha_s^3)$:

$$\Lambda = 233 \pm 30 \,\text{MeV}$$

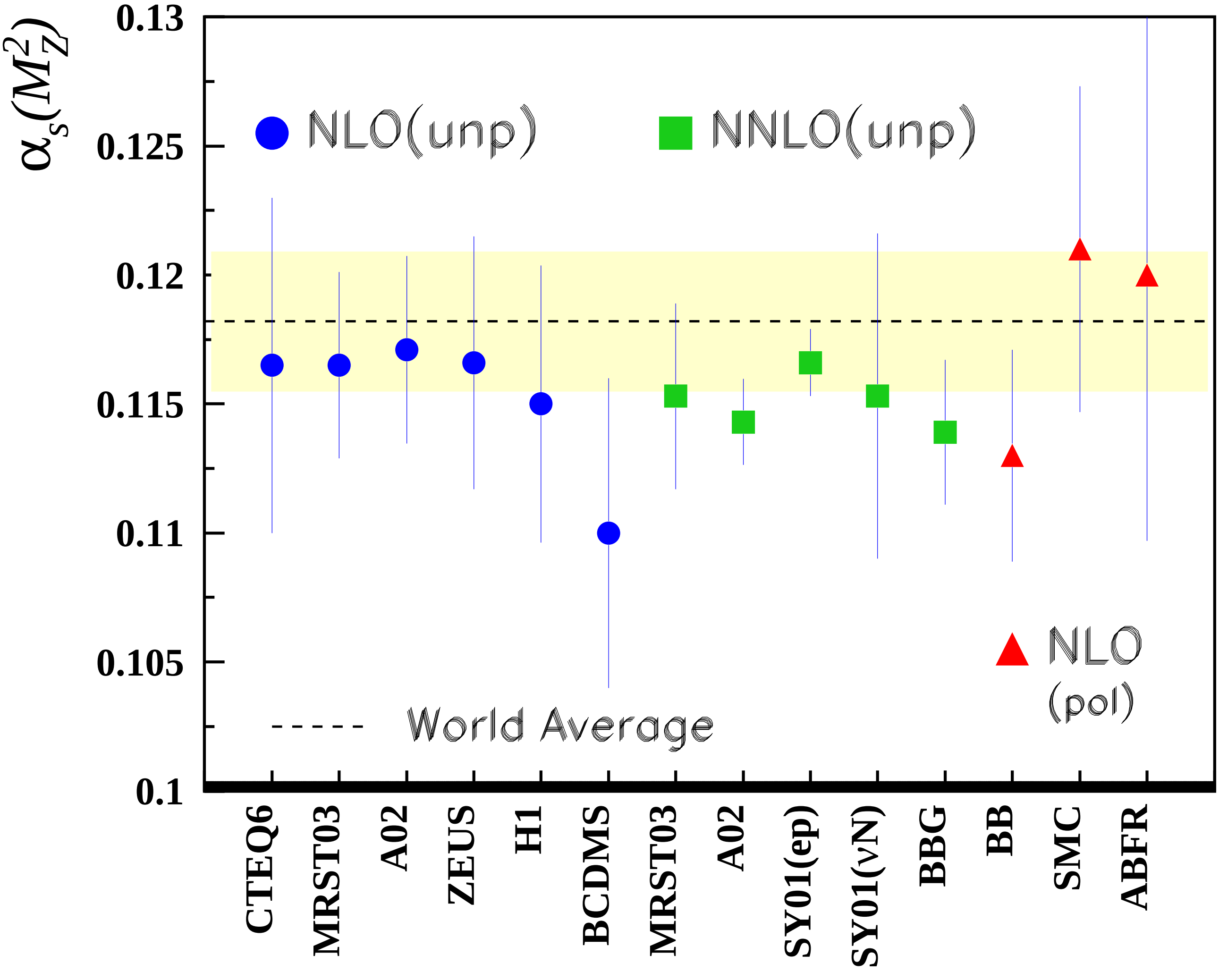
Alpha Collab: $N_f = 2$ Lattice; non-pert. renormalization

$$\Lambda = 245 \pm 16 \pm 16 \,\text{MeV}$$

QCDSF Collab: $N_f = 2$ Lattice, pert. reno.

$\Lambda = 249 + 13 + 13 / - 8 - 17 \,\text{MeV}$ also other collab., (cf. PDG).

DIS: $\alpha_s(M_Z^2)$



9. Future Avenues

The Future is always bright.

HERA:

- Collect high luminosity for $F_2(x, Q^2)$, $F_2^{c\bar{c}}(x, Q^2)$, $g_2^{c\bar{c}}(x, Q^2)$, and measure $h_1(x, Q^2)$.
- Measure : $F_L(x, Q^2)$. This is a key-question for HERA.

RHIC & LHC:

- Improve constraints on gluon and sea–quarks: polarized and unpolarized.

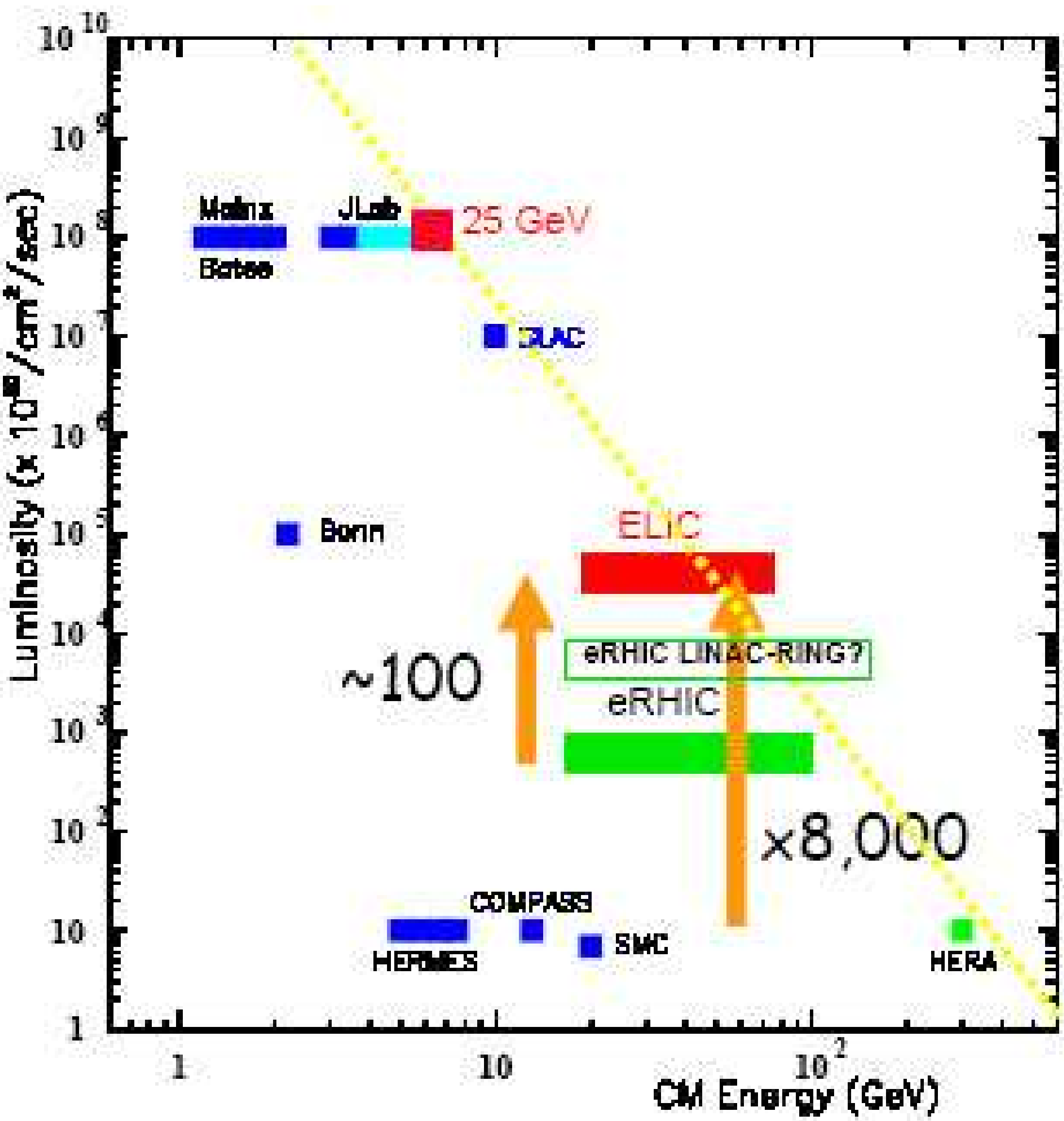
JLAB:

- High precision measurements in the large x domain at unpolarized and polarized targets; supplements HERA's high precision measurements at small x.

ELIC:

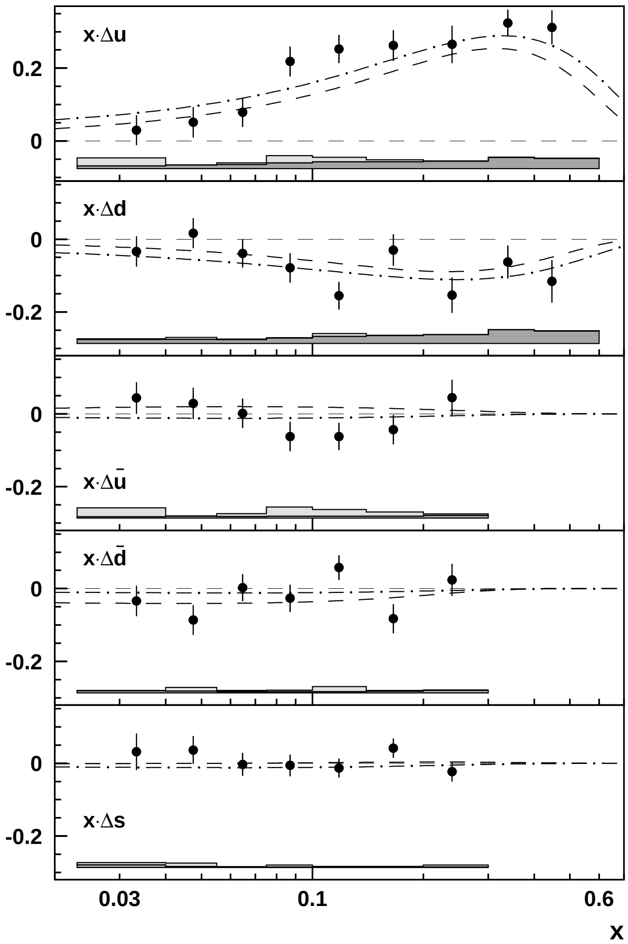
- High precision measurements in the medium x domain; both unpolarized and polarized

THE QUEST FOR LARGE LUMINOSITY !

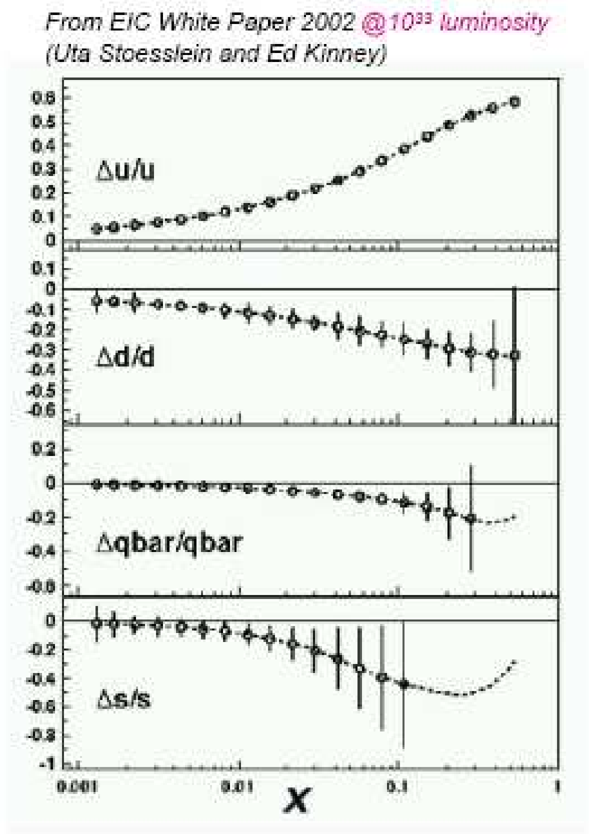


.... allows very precise measurements

Example : Flavor Separation of polarized PDF's



HERMES



EIC

- What is the correct value of $\alpha_s(M_z^2)$? $\overline{\mathrm{MS}}$-analysis vs. scheme-invariant evolution helps. Compare non–singlet and singlet analysis; careful treatment of heavy flavor.[Theory & Experiment]
- Flavor Structure of Sea-Quarks: More studies needed.[All Experiments]
- Revisit polarized data upon arrival of the 3-loop anomalous dimensions; NLO heavy flavor contributions needed.[Theory]
- QCD at Twist 3: $g_2(x, Q^2)$, semi–exclusive Reactions [High Precision polarized experiments, JLAB, EIC]
- Comparison with Lattice Results: α_s, Moments of Parton Distributions, Angular Momentum.
- Calculation of more hard scattering reactions at the 3–loop level: ILC, LHC
- Further perfection of the mathematical tools:
  $\Longrightarrow$ Algorithmic simplification of Perturbation theory in higher orders.
- Even higher order corrections needed ?