Simulation Algorithm and Benchmarking Results

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1. Introduction

Coherent Synchrotron RadiationEffect in Bends

- **Lienard-Wiechert Field Generated by a Single Particle**

$$\vec{E}_0(\vec{r}, t) = e\left(\frac{\vec{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\vec{n})^3 R^2}\right)_{ret} + \frac{e}{c}\left(\frac{\vec{n}\times\{(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{\gamma^2(1-\vec{\beta}\cdot\vec{n})^3 R}\right)_{ret}$$

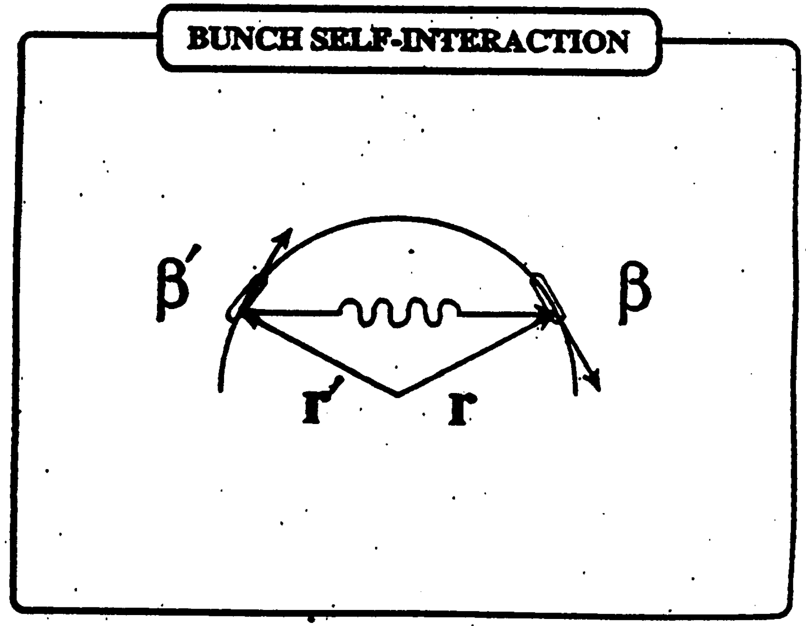
$$\vec{B}_0(\vec{r}, t) = (\hat{n}\times\vec{E}_0)_{ret}$$

Retardation : $t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$

- **Collective Self-Field for a Bunch**

$$\vec{F}(r, t) = \int e(\vec{E}_0 + \vec{\beta}\times\vec{B}_0)\, n(\vec{s}')\, d\vec{s}'$$

- **Cause Emittance Degradation**



General Problem for Simulation to Solve

- The dynamics for an electron in the bunch is governed by

$$\frac{d(\gamma\, m_e \dot{\vec{v}})}{dt} = e(\vec{E} + \vec{\beta} \times \vec{B}) \qquad \begin{cases} \vec{E} = \vec{E}^{\mathrm{ext}} + \vec{E}^{\mathrm{self}} \\ \vec{B} = \vec{B}^{\mathrm{ext}} + \vec{B}^{\mathrm{self}} \end{cases}$$

- The bunch self-interaction fields depend on the history of the bunch charge distribution and current density

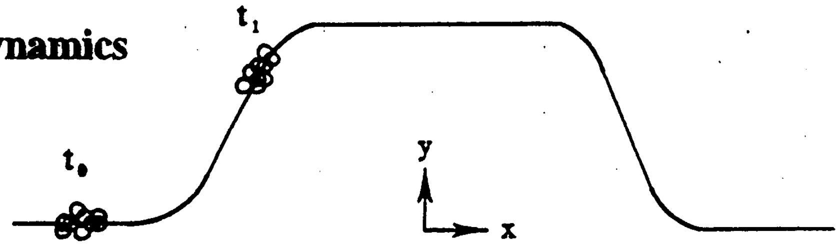
$$\vec{E}^{\mathrm{self}} = -\vec{\nabla}\Phi - \frac{\partial \vec{\mathrm{A}}}{c\partial \mathrm{t}}, \quad \overline{B}^{\mathrm{self}} = \vec{\nabla} \times \vec{A} \qquad \begin{cases} \Phi(\vec{r},t) = \int d\vec{r}' \dfrac{\rho(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{|r - r'|} \\ \vec{A}(r,t) = \int d\vec{r}' \dfrac{J(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{|\vec{r} - \vec{r}'|} \end{cases}$$

2. Simulation Algorithm

Macroparticle Model

- **Transverse and longitudinal bunch dynamics**

(free space)



- **Bunch is simulated by a set of macroparticles**

— Macros : 2 D (longitudinal-transverse) round Gaussian disc (no vertical extent)

— Single macro density distribution :

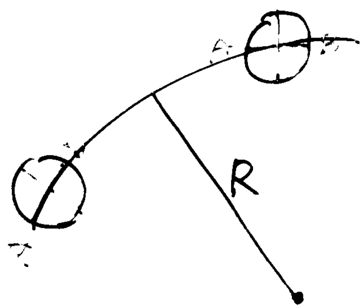
$$n_m(\mathbf{r} - \mathbf{r}_0(t)) = \frac{1}{2\pi\sigma_m^2}\,\mathrm{Exp}\left[-\frac{(x - x_0(t))^2 + (y - y_0(t))^2}{2\sigma_m^2}\right]$$

σ_m : rms size of each macro $\qquad$ $\mathbf{r}_0(t)$: macro centroid vector at time t

— Bunch charge distribution and current density

$$\begin{cases} \rho(\mathbf{r}, t) = q_m \sum n_m(\mathbf{r} - \mathbf{r}_0^{(i)}(t)) \\ \mathbf{J}(\mathbf{r}, t) = q_m \sum \boldsymbol{\beta}_0^{(i)}(t)\, n_m(\mathbf{r} - \mathbf{r}_0^{(i)}(t)) \end{cases}$$

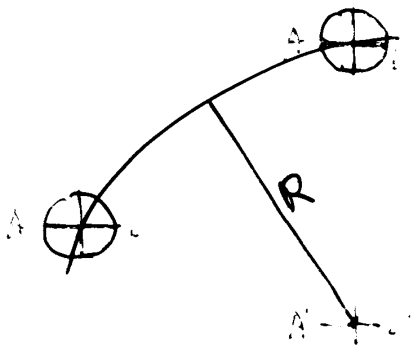
•Caution on the Motion of Macroparticle



If the whole macroparticle rotates about the center of design orbit, then the outer part of the macroparticle will have speed bigger than c:

$$\beta_0(1+\frac{x}{R}) > c \qquad \beta_0 = \frac{v}{c}$$

•Present Treatment



At each step, the whole macroparticle moves with its centroid *translationally* without rotation.
This is equivalent to different part of the macroparticle moves about its own center of orbit.

•Choice of Simulation Parameters σ_m and N_m

—— so as to best represent the actual bunch

For Gaussian bunch: $\sigma_s^{\text{eff}} = \sqrt{\sigma_s^2 + \sigma_m^2}$

(for $\sigma_{\text{m}} = \sigma_s / 3$, $\sigma_s^{\text{eff}} \approx 1.05\sigma_s$)

Round macroparticle: $\sigma_x^{\text{eff}} = \sqrt{\sigma_x^2 + \sigma_m^2}$

(for $\sigma_x << \sigma_s$, this may give artificial effective transverse size to the bunch)

N_m should be chosen to ensure the overlap of macroparticles for suppression of shot noise

However, the finite macrosize could suppress fine structures,
So only suitable to study dynamics for smooth bunch distribution.

CSR Force Computation

- Using macroparticle model,

$$\begin{cases} \rho(\vec{r},t) = q_m \sum_{i=1}^{N_m} n_m(\vec{r} - \vec{r}_0^{(i)}(t)) \\ \vec{J}(\vec{r},t) = q_m \sum_{i=1}^{N_m} \vec{\beta}_0^{(i)}(t)\, n_m(\vec{r} - \vec{r}_0^{(i)}(t)) \end{cases}$$

- The potentials become

$$\begin{cases} \Phi(\vec{r},t) = q_m \sum_{i=1}^{N_m} \int \frac{n_m(\vec{r}' - \vec{r}_0^{(i)}(t'))}{|\vec{r} - \vec{r}'|} d\vec{r}' \\ A(\vec{r},t) = q_m \sum_{i=1}^{N_m} \int \frac{\vec{\beta}_0^{(i)}(t') n_m(\vec{r}' - \vec{r}_0^{(i)}(t'))}{|\vec{r} - \vec{r}'|} d\vec{r}' \end{cases}$$

(with $t' = t - \frac{|\vec{r} - \vec{r}'|}{c}$)

- Self-Fields

$$\vec{E}^{\text{self}} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{c\partial t}, \quad \vec{B}^{\text{self}} = \vec{\nabla} \times \vec{A}$$

CSR Force Computation

- **Wakefield obtained by applying macroparticle model**

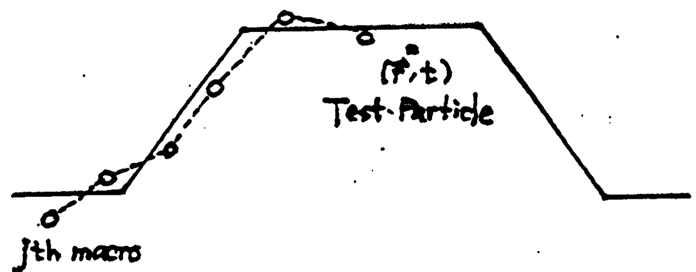
$$\mathbf{E}(\mathbf{r}, t) = \sum_j \mathbf{E}^{(j)}(\mathbf{r}, t), \quad B_z(\mathbf{r}, t) = \sum_j B_z^{(j)}(\mathbf{r}, t)$$

$$\mathbf{E}^{(j)}(\mathbf{r}, t) = q_m \int \frac{d^2\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \left\{ \frac{\mathbf{r}' - \mathbf{r}_0^{(j)}(t')}{\sigma_m} - \frac{\boldsymbol{\beta}_0^{(j)}(t')}{c}\sigma_m - \boldsymbol{\beta}_0^{(j)}(t')\left[\frac{\mathbf{r}' - \mathbf{r}_0^{(j)}(t')}{\sigma_m} \cdot \boldsymbol{\beta}_0^{(j)}(t')\right] \right\} n_m(\mathbf{r} - \mathbf{r}_0^{(j)}(t'))$$

$$\mathbf{B}^{(j)}(\mathbf{r}, t) = q_m \int \frac{d^2\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \left[\boldsymbol{\beta}_0^{(j)}(t') \times \frac{\mathbf{r}' - \mathbf{r}_0^{(j)}(t')}{\sigma_m}\right] \} n_m(\mathbf{r} - \mathbf{r}_0^{(j)}(t'))$$

—Each pair interaction is a 2 D integration over area surrounding jth macro' s previous path

- **Interpretation**



If we divide jth macro into grid, each grid emits photon (which reaches r at t) at different r', t'.

- **Numerical Implementation : 2 D Simpson Rule**

Subtleties for Correctly Computing the Integrations

- **Treatment of Retardation**

  —Evaluate the integrand at retarded time

  —Computing $(\mathbf{r}_0(t'), \mathbf{v}_0(t'))$ of macro centroids by interpolation using recorded histroy of $(\mathbf{r}_0(t_k), \mathbf{v}_0(t_k))$ at discrete time step

- **Singularity Removal**

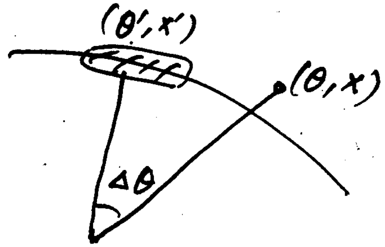
  —Integration by parts

- **Localized Behavior of Retardation Relation**

  —Separate the integrand into long range and short range behaving functions

  —Remove singularity for short range behaving function

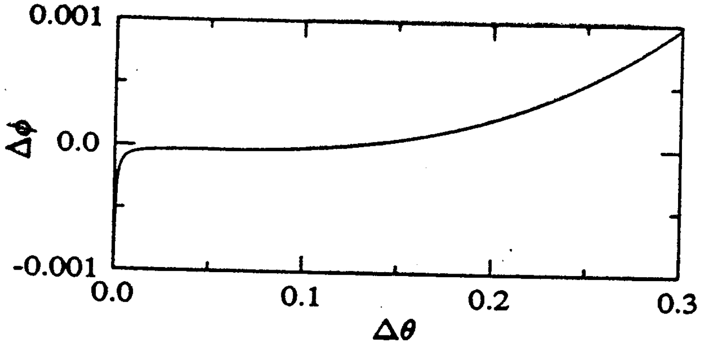
Local and Long Range Behavior of Retardation Relation



$$\Phi(s,x,t) = \int \frac{\rho(s',x',t')}{|\vec{r}-\vec{r}'|}(R+x')d\theta' dx' = \int \frac{\rho(s-\Delta s,x',t')}{|\vec{r}-\vec{r}'|}(R+x')d\theta' dx'$$

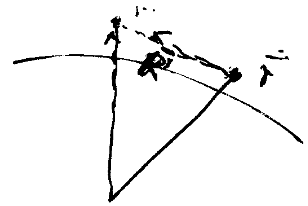
$$\Delta s = R\Delta\theta - \beta|\vec{r}-\vec{r}'| = R\Delta\theta - \beta\sqrt{(R+x)(R+x')(2\mathrm{Sin}(\tfrac{\Delta\theta}{2}))^2 + (x-x')^2}$$

$$\Delta\varphi = \Delta s / R \approx \frac{\Delta\theta^3}{24} - \Delta\theta\frac{x+x'}{2R} - \frac{(x-x')^2}{2R^2\Delta\theta} \qquad (\text{for } \Delta\theta^2 >> \frac{\sigma_\perp}{R})$$



Local and long range behavior of $\Delta\phi$ vs. $\Delta\theta$ for $x/R = 0.001$.

Removal of Singularity



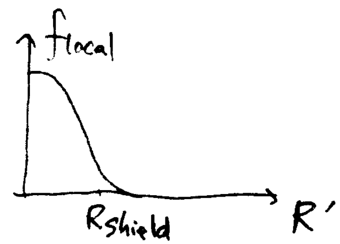
$$\Phi(\vec{r},t)=\sum_{i=1}^{N_m}\Phi^{(i)}(\vec{r},t),\quad \text{with}\quad \Phi^{(i)}(\vec{r},t)=q_m\int\frac{n_m(\vec{r}'-\vec{r}_0^{(i)}(t-|\vec{r}-\vec{r}'|/c))}{|\vec{r}-\vec{r}'|}d\vec{r}'$$

$$\text{Let } \vec{R}'=\vec{r}'-\vec{r},\quad \Phi^{(i)}(\vec{r},t)=q_m\int\frac{n_m(\vec{R}'+\vec{r}-\vec{r}_0^{(i)}(t-R'/c))}{R'}d\vec{R}'=\Phi^{(i)}_{\text{local}}+\Phi^{(i)}_{\text{long}}$$

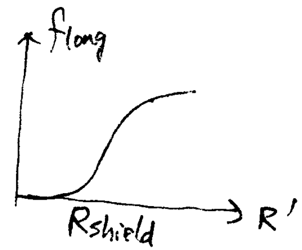
$$\Phi^{(i)}_{\text{local}}=q_m\int\frac{d\vec{R}'}{R'}n_m(\vec{r}+\vec{R}'-\vec{r}_0^{(i)}(t-R'/c))\,\underbrace{\text{Exp}\left(-\frac{R'^2}{2R^2_{\text{shield}}}\right)}_{f_{local}}$$

Singularity

$f_{local} \Rightarrow$ select local range



$$\Phi^{(i)}_{\text{long}}=q_m\int\frac{d\vec{R}'}{R'}n_m(\vec{r}+\vec{R}'-\vec{r}_0^{(i)}(t-R'/c))\,\underbrace{\left[1-\text{Exp}\left(-\frac{R'^2}{2R^2_{\text{shield}}}\right)\right]}_{f_{long}}$$



$f_{long} \Rightarrow$ select long range

Singularity Removal for $\Phi_{\text{local}}^{(i)}$

- **After integration by parts,**

$$\Phi_{\text{local}}^{(i)} = q_m \int d\vec{R}' n_m(\vec{r} + \vec{R}' - \vec{r}_0^{(i)}(t - R'/c)) f_{\text{local}}(R')$$
$$\times \left\{ \frac{r + R' - r_0^{(i)}(t - R'/c)}{\sigma_m^2} \cdot \left[\frac{\vec{R}'}{R'} + \vec{\beta}_0^{(i)}(t - R'/c) \right] + \frac{R'}{R_{\text{shield}}^2} \right\}$$

no more singularity!

- **Similarly, singularity can be removed for vector potential**

$$\vec{A} = \sum_{i=1}^{N_m} \vec{A}^{(i)}, \qquad \vec{A}^{(i)} = \vec{A}_{\text{local}}^{(i)} + \vec{A}_{\text{long}}^{(i)}$$

Final CSR force computed is insensitive to choice of R_{shield}

Self-Interaction Field

The self-interaction field can now be computed using Eq. (2), which gives

$$\mathbf{E}^{\mathrm{self}} = \mathbf{E}^{\mathrm{self}}_{\mathrm{local}} + \mathbf{E}^{\mathrm{self}}_{\mathrm{long}} \qquad \mathbf{B}^{\mathrm{self}} = \mathbf{B}^{\mathrm{self}}_{\mathrm{local}} + \mathbf{B}^{\mathrm{self}}_{\mathrm{long}}$$

Taking derivatives of Eqs. (14) and (16) according to Eq. (2), and changing the integral $\int d\mathbf{R}'$ back to $\int d\mathbf{r}'$, we get

$$\boxed{\mathbf{E}^{\mathrm{self}} = \frac{eq_m}{\sigma_m^2}\sum_i \int d\mathbf{r}' n_m(\mathbf{r}_{sc}^{(i)})\, \mathbf{e}^{(i)}(\mathbf{r}'),\ \mathbf{B}^{\mathrm{self}} = \frac{eq_m}{\sigma_m^2}\sum_i \int d\mathbf{r}' n_m(\mathbf{r}_{sc}^{(i)})\, \mathbf{b}^{(i)}(\mathbf{r}'),}$$

where $\mathbf{r}_{sc}^{(i)} = \mathbf{r}' - \mathbf{r}_0^{(i)}(t')$, $\hat{\mathbf{r}}_{sc} = \mathbf{r}_{sc}^{(i)}/\sigma_m$, and $\mathbf{e}^{(i)}(\mathbf{r}')$ and $\mathbf{b}^{(i)}(\mathbf{r}')$ are dimensionless vectors:

$$\begin{aligned}
\mathbf{e}^{(i)}(\mathbf{r}') &= \mathbf{e}_1 f_{\mathrm{local}} - \mathbf{e}_2\, \sigma_m f'_{\mathrm{local}} + \mathbf{e}_2 \frac{\sigma_m}{|\mathbf{r}-\mathbf{r}'|} f_{\mathrm{long}} \\
\mathbf{b}^{(i)}(\mathbf{r}') &= \mathbf{b}_1 f_{\mathrm{local}} - \mathbf{b}_2\, \sigma_m f'_{\mathrm{local}} + \mathbf{b}_2 \frac{\sigma_m}{|\mathbf{r}-\mathbf{r}'|} f_{\mathrm{long}}
\end{aligned}$$

with $f'_{\mathrm{local}} = df_{\mathrm{local}}(R')/dR'$. For $\mathbf{n} = (\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$, $\mathbf{n}_\beta = \mathbf{n} + \beta_0^{(i)}(t')$, and $a_\beta = \dot{\beta}_0^{(i)}\sigma_m/c$, we have

$$\begin{aligned}
\mathbf{e}_1 &= \hat{\mathbf{r}}_{sc}(\hat{\mathbf{r}}_{sc}\cdot\mathbf{n}_\beta) - \mathbf{n}_\beta - \mathbf{a}_\beta[\hat{\mathbf{r}}_{sc}\cdot\mathbf{n}_\beta + \hat{\mathbf{r}}_{sc}\cdot\beta_0^{(i)}] \\
&\quad + \beta_0^{(i)}[\beta_0^{(i)}\cdot\mathbf{n}_\beta - (\hat{\mathbf{r}}_{sc}\cdot\mathbf{n}_\beta)(\hat{\mathbf{r}}_{sc}\cdot\beta_0^{(i)})], \\
\mathbf{b}_1 &= (\mathbf{n}_\beta\times\beta_0^{(i)}) - (\hat{\mathbf{r}}_{sc}\cdot\mathbf{n}_\beta)(\hat{\mathbf{r}}_{sc}\times\beta_0^{(i)}) + \mathbf{a}_\beta\times\hat{\mathbf{r}}_{sc}, \\
\mathbf{e}_2 &= \hat{\mathbf{r}}_{sc} - \mathbf{a}_\beta - \beta_0^{(i)}(\hat{\mathbf{r}}_{sc}\cdot\beta_0^{(i)}), \\
\mathbf{b}_2 &= \beta_0^{(i)}\times\mathbf{r}_{sc}^{(i)}
\end{aligned}$$

for $\beta_0^{(i)}$ and $\dot{\beta}_0^{(i)}$ be evaluated at t'. The above formulae are derived for constant σ_m. If the size of the macroparticles is allowed to change with time, then $d\sigma_m/dt$ should be taken into account.

Numerical Procedure for Field Calculation

$$\begin{cases} \vec{E}^{\text{self}}(\vec{r},t) = q_m \sum_i \int d\vec{r}' \, \vec{E}_{\text{integrand}}(\vec{r}-\vec{r}', \vec{r}'-\vec{r}_0^{(i)}(t'), \vec{\beta}_0^{(i)}(t'), \dot{\vec{\beta}}_0^{(i)}(t')) \\ \vec{B}^{\text{self}}(\vec{r},t) = q_m \sum_i \int d\vec{r}' \, \vec{B}_{\text{integrand}}(\vec{r}-\vec{r}', \vec{r}'-\vec{r}_0^{(i)}(t'), \vec{\beta}_0^{(i)}(t'), \dot{\vec{\beta}}_0^{(i)}(t')) \end{cases}$$

Given $(\vec{r},t)$,

(1) For each $\vec{r}'$, evaluate $\vec{E}_{\text{integrand}}$ and $\vec{B}_{\text{integrand}}$

(2) Set integration range for $\vec{r}'$ properly

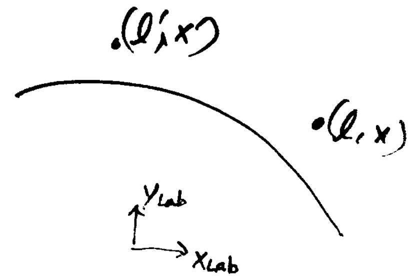
(3) Adaptive Simpson rule for integration

(1) Given $(\vec{r}, t)$, for each $\vec{r}'$, evaluate $(\vec{E}, \vec{B})_{\text{integrand}}$ as a function of

$$(|\vec{r}-\vec{r}'|, \vec{r}'-\vec{r}_0^{(i)}(t'), \vec{\beta}_0^{(i)}(t'), \dot{\vec{\beta}}_0^{(i)}(t'))$$

- There is a mapping routine to map curvilinear coordinates to absolute lab coordinates

$$\vec{r}: \quad (l,x) \rightarrow (x_{Lab}, y_{Lab})$$
$$\vec{r}': \quad (l',x') \rightarrow (x'_{Lab}, y'_{Lab})$$
$$|\vec{r}-\vec{r}'| = \sqrt{(x_{Lab}-x'_{Lab})^2 + (y_{Lab}-y'_{Lab})^2}$$



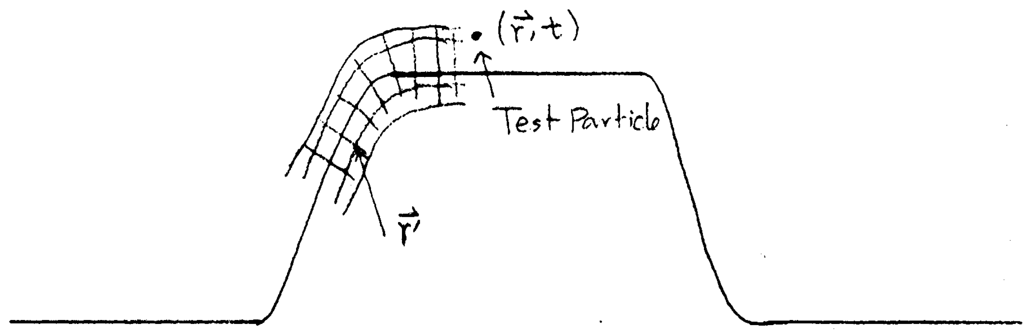
- Retardation time is calculated by $t' = t - |\vec{r}-\vec{r}'|/c$

$\vec{r}_0^{(i)}(t'), \vec{\beta}_0^{(i)}(t')$ for the ith macroparticle can be linearly interpolated from recorded history for macro centroids at discrete times:

$$\text{For} \quad t' \in (t_k, t_{k+1}), \quad \vec{r}_0^{(i)}(t') = \vec{r}_0^{(i)}(t_k) + \frac{\vec{r}_0^{(i)}(t_{k+1}) - \vec{r}_0^{(i)}(t_k)}{\Delta t}(t'-t_k)$$

etc.

Retardation Treatment



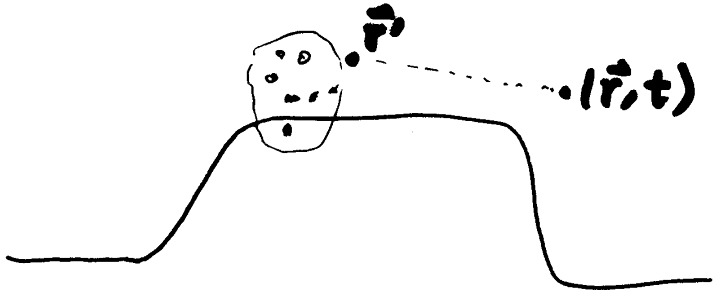
$$(\varphi, \mathbf{A})(\mathbf{r}, t) = \sum_i \int d^2\mathbf{r}' \left(1, \boldsymbol{\beta}_0^{(i)}(t')\right) \frac{n_m\left(\mathbf{r}' - \mathbf{r}_0^{(i)}(t')\right)}{|\mathbf{r} - \mathbf{r}'|}$$

For each $\mathbf{r}'$, one needs to find $\mathbf{r}_0^{(i)}$, $\boldsymbol{\beta}_0^{(i)}$ at $t' = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \Big|$.

This can be interpolated from the recorded history at discrete times $\mathbf{r}_0^{(i)}(t^k)$, $\boldsymbol{\beta}_0^{(i)}(t^k)$

- Macroparticle size

Given $(\vec{r}, t)$, and $\vec{r}'$, found $t' = t - |\vec{r} - \vec{r}'|/c$, and then the macro-centroids at retarded time $\vec{r}_0^{(i)}(t')$



The macrosize is chosen to be 1/3 of $\sigma_s(t')$:

$$\sigma_{\mathrm{m}}(t') = \sigma_s(t')/3$$

to best represent the charge distribution at t' on $\vec{r}'$.

Field Calculation: (2) Set Integration Range

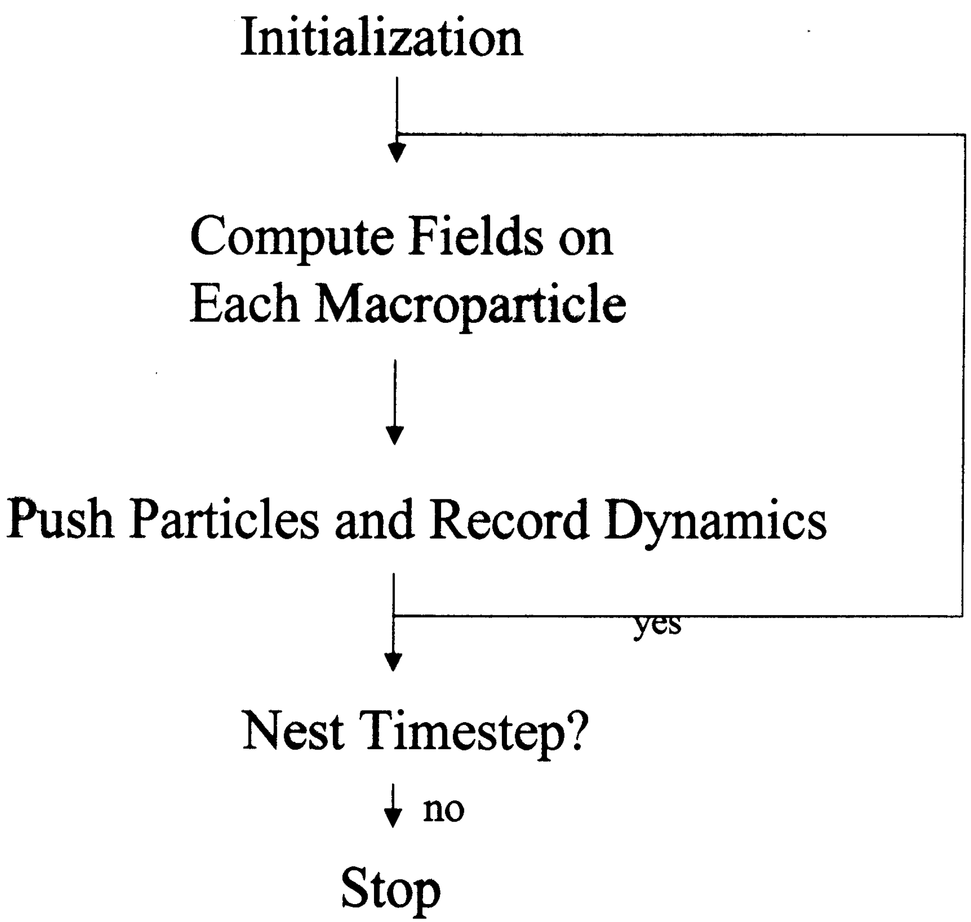
— over the bunch

— neighborhood region for $\vec{E}_{\text{local}}, \vec{B}_{\text{local}}$ (finer mesh)

— far back region $\left(\frac{\Delta l}{R}\right)^2 >> \frac{\sigma_\perp}{R}$

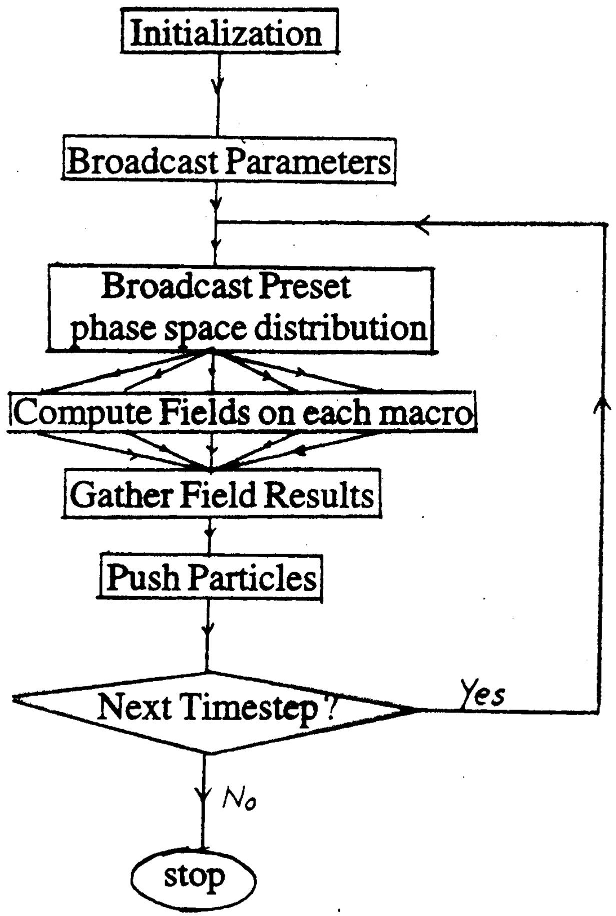
Here, a coarse grid calculation of field integrand is done to estimate the integration region.

Layout of Simulation



Parallel Computation

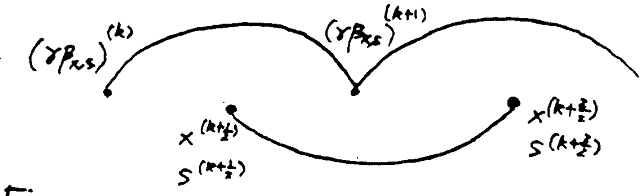
Flow Chart on Usage of MPI



- **Initialization** →
  - Read Beam line
  - Read Beam Parameters
  - Initialize Beam ↓

InitLine
InitSlope
InitRandm
InitPhase0

Leap-Frog Scheme



- **Equation of Motion in Circular Orbit**

$$\begin{cases} \dfrac{d}{dt}(\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 = e\left(E_r - \dfrac{r\dot{\theta}}{c} B_z\right), = F_r \\[2ex] \dfrac{d}{dt}(\gamma m r \dot{\theta}) + \gamma m \dot{r} \dot{\theta} = e\left(E_\theta + \dfrac{\dot{r}}{c} B_z\right) = F_\theta \end{cases}$$

- **Numerical Expression** (Design orbit radius ρ)

$$\begin{cases} \dfrac{(\gamma\beta_x)^{(k+1)} - (\gamma\beta_x)^{(k)}}{c\Delta t} = \left[\tilde{E}_x - \beta_s\left(\dfrac{\gamma\beta_s}{r} - \dfrac{\gamma_0\beta_0}{\rho} + \tilde{B}_z\right)\right]^{(k+\frac{1}{2})} \\[2ex] \dfrac{(\gamma\beta_s)^{(k+1)} - (\gamma\beta_s)^{(k)}}{c\Delta t} = \left[\tilde{E}_s + \beta_x\left(\dfrac{\gamma\beta_s}{r} - \dfrac{\gamma_0\beta_0}{\rho} + \tilde{B}_z\right)\right]^{(k+\frac{1}{2})} \end{cases}, \quad \begin{cases} x^{(k+\frac{3}{2})} = x^{(k+\frac{1}{2})} + \beta_x^{(k+1)} c\Delta t \\[2ex] s^{(k+\frac{3}{2})} = s^{(k+\frac{1}{2})} + \dfrac{\rho\,\beta_s^{(k+1)}}{r} c\Delta t \end{cases}$$

with

$$\begin{cases} x = r - \rho \\ s = \rho\theta \end{cases}, \quad \begin{cases} \gamma\beta_x = \gamma\dot{r}/c \\ \gamma\beta_s = \gamma r\dot{\theta}/c \end{cases} \qquad \begin{matrix} \gamma^2 = (\gamma\beta_x)^2 + (\gamma\beta_s)^2 + 1 \\ \tilde{E}_{x,s}[1/m] = \dfrac{e}{mc^2} E_{x,s}^{CSR} \end{matrix}$$

- **Conservation of Energy**

$$\frac{\gamma^{(k+1)} - \gamma^{(k)}}{c\Delta t} = \left[\tilde{E} \cdot \beta\right]^{(k+\frac{1}{2})}$$

Constraints on "Self-Consistency" due to Macroparticle Model

- The brute force integration for CSR force computation is CPU intensive. As a result, N_m is typically 250-500.

- To minimize shot noise, the macroparticles are given finite sizes, typically $\sigma_m \approx \sigma_s / 3$.

- The emergence of fine structures and microbunching is suppressed artificially by the finite macro sizes. $\Rightarrow$ Only suitable to study dynamics for smooth distribution.

- 2D simulation $\Rightarrow$ space charge is not properly simulated.

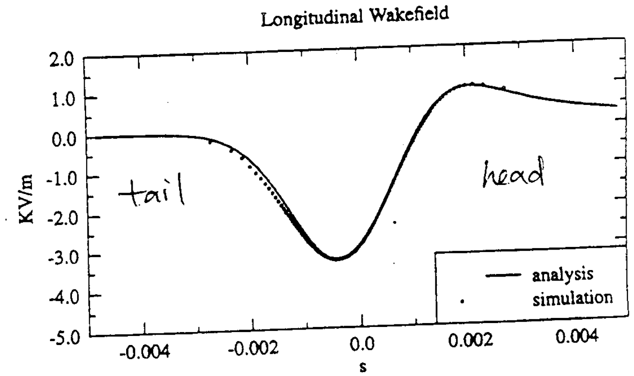
3. Benchmark with Analysis

- Longitudinal CSR Force (Steady-State Gaussian-Line-Bunch)
- Effective Transverse Collective Force (Gaussian-line-Bunch)
- CSR Force for a Compressed Bunch (nonlinear energy chirp)

Collective Self-Interaction Forces (free-space, steady-state)

● **Longitudinal Collective Force (Gaussian-line-bunch)**

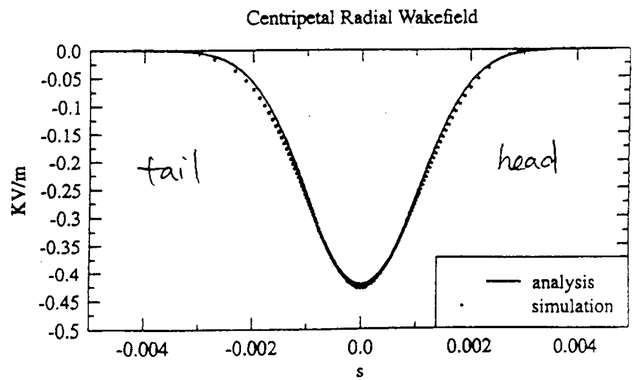
$$e\,\frac{\partial(\varphi-\beta\bullet A)}{c\,\partial t} = \frac{2Ne^2}{(3R^2\sigma_s^4)^{1/3}}\,\frac{\partial}{\partial\varphi}\int_0^\infty \frac{d\varphi_1}{\varphi_1^{1/3}}\,\lambda(\varphi-\varphi_1)$$



$\sigma_s = 1\,mm$
$R = 1\,m$
$Q = 60\,pC$

● **Transverse Collective Force**

$$-e\,\frac{\partial(\varphi-\beta\bullet A)}{\partial r} = \frac{-2Ne^2\,\lambda(\varphi)}{R\,\sigma_s}$$

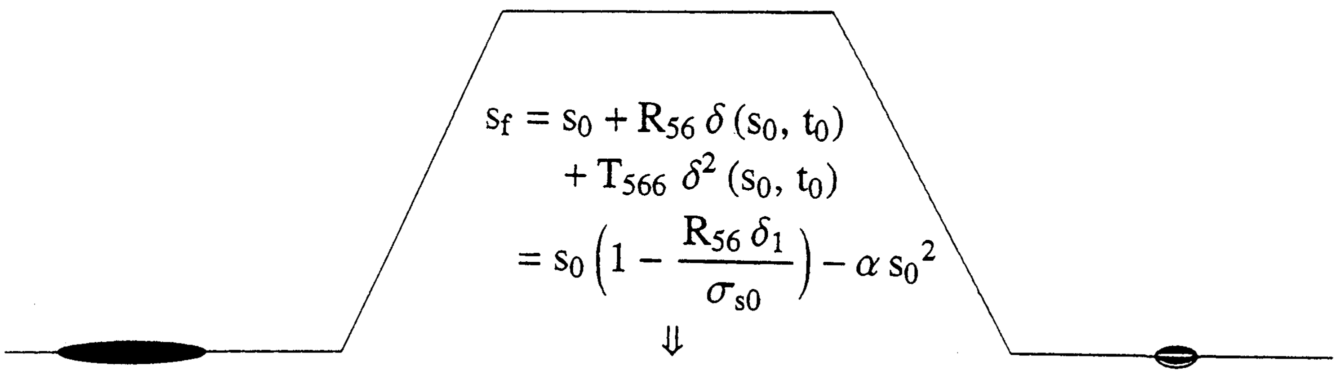


Longitudinal Charge Distribution for a Compressed Bunch

Assuming 2 nd order nonlinearity in $\begin{cases}\text{initial energy correlation} \\ \text{longitudinal optics}\end{cases}$

- $\boldsymbol{\delta_{un} = 0}$

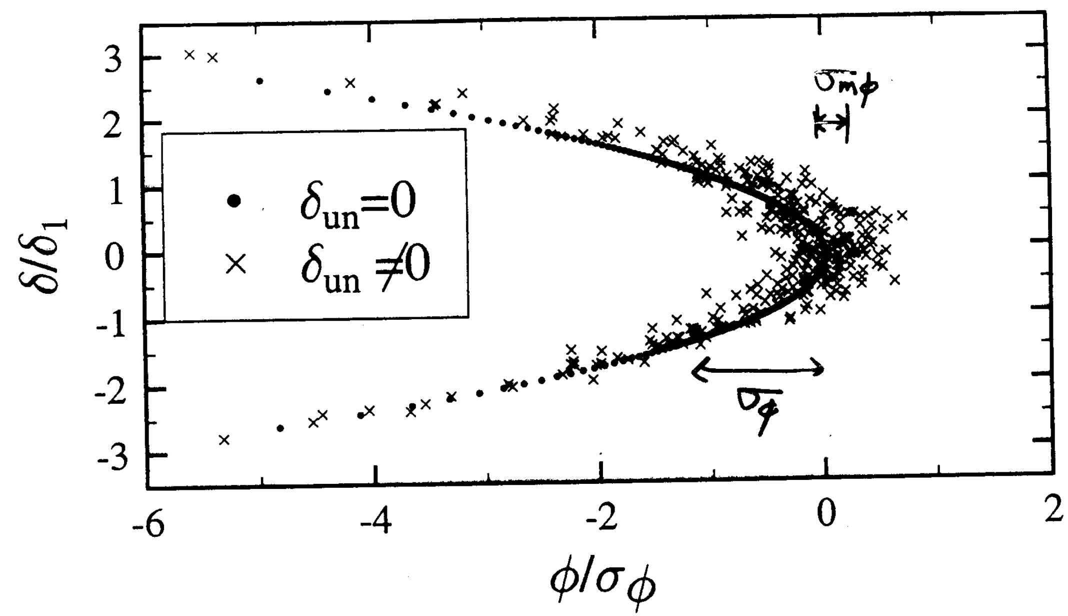
$$\delta(s_0, t_0) = -\delta_1 \left(\frac{s}{\sigma_{s0}}\right) - \delta_2 \left(\frac{s}{\sigma_{s0}}\right)^2 + \delta_{un} \qquad\qquad s_f = -\left(\frac{\sigma_{sf}}{\sqrt{2}\,\delta_1{}^2}\right)\delta^2 + R_{56}\,\delta_{un}$$



$$\begin{aligned} s_f &= s_0 + R_{56}\,\delta(s_0, t_0) \\ &\quad + T_{566}\,\delta^2(s_0, t_0) \\ &= s_0\left(1 - \underbrace{\frac{R_{56}\,\delta_1}{\sigma_{s0}}}_{\Downarrow\ 0}\right) - \alpha\, s_0{}^2 \end{aligned}$$

when fully compressed

$$n(s_0, t_0)\, \frac{1}{\sqrt{2\pi}\,\sigma_{s0}}\ \mathrm{Exp}\left[\frac{-s_0{}^2}{2\,\sigma_{s0}{}^2}\right] \qquad\qquad n(s_f, t_f) = \frac{1}{\sqrt{2\pi}\,\sigma_{sf}}\ \frac{\mathrm{Exp}\left[s_f / \sqrt{2}\,\sigma_{sf}\right]}{\sqrt{-s_f / \sqrt{2}\,\sigma_{sf}}}$$



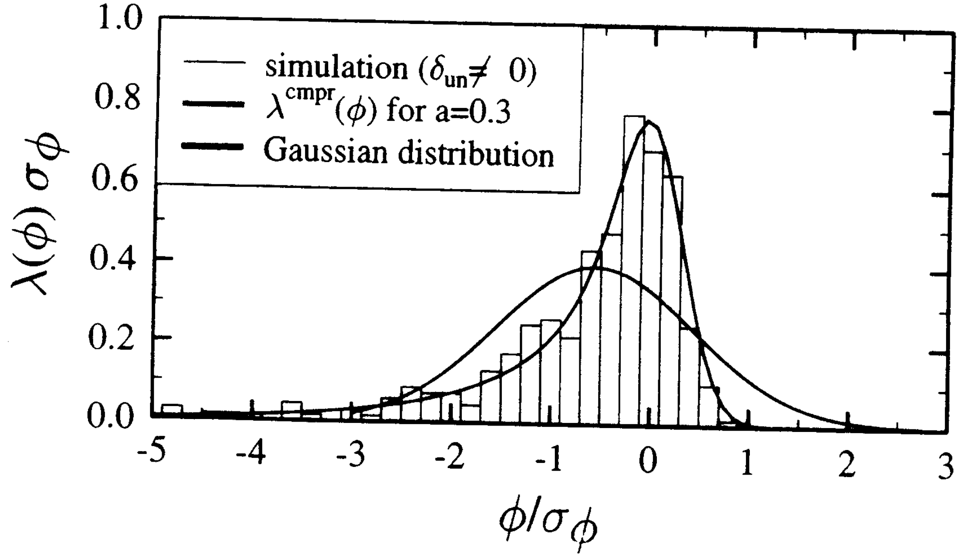
$$\text{Let } a = \frac{\sigma_{m\phi}}{\sigma_\phi} = \frac{R_{56}\,\delta_{un}^{rms}}{\sigma_{sf}}$$

Longitudinal Density Dsitribution for a Compressed Bunch

$$\text{For } u = \frac{a}{\sqrt{2}} + \frac{\phi}{\sigma_{m\phi}}, \quad \theta(u) = \begin{cases} 1 & (u<0) \\ -1 & (u>0) \end{cases}$$

$$\lambda^{cmpr}(\phi) = \frac{2^{1/4}}{4\sigma_\phi \sqrt{a}} \, \mathrm{Exp}\left(-\frac{\phi^2}{2a^2\sigma_\phi{}^2} + \frac{u^2}{4}\right)\sqrt{|u|}\left(I_{1/4}\left(\frac{u^2}{4}\right) + \theta(u)\, I_{1/4}\left(\frac{u^2}{4}\right)\right)$$

Analytical/Numerical Results of Longitudinal Phase Space



CSR Force for a Compressed Bunch ($\delta_{u} \neq 0$)

■ **Longitudinal CSR Force**

$$F_{\theta}^{cmpr}(\phi) = \int_{-\infty}^{\infty} F_{\theta 0}^{cmpr}(\phi_1)\,\lambda_m(\phi-\phi_1)\,d\phi_1 \quad = F_g \frac{2^{1/4}}{\sqrt{2\pi}\,a^{5/6}}\; f\left(\frac{\phi}{a\,\sigma_\phi};a\right)$$

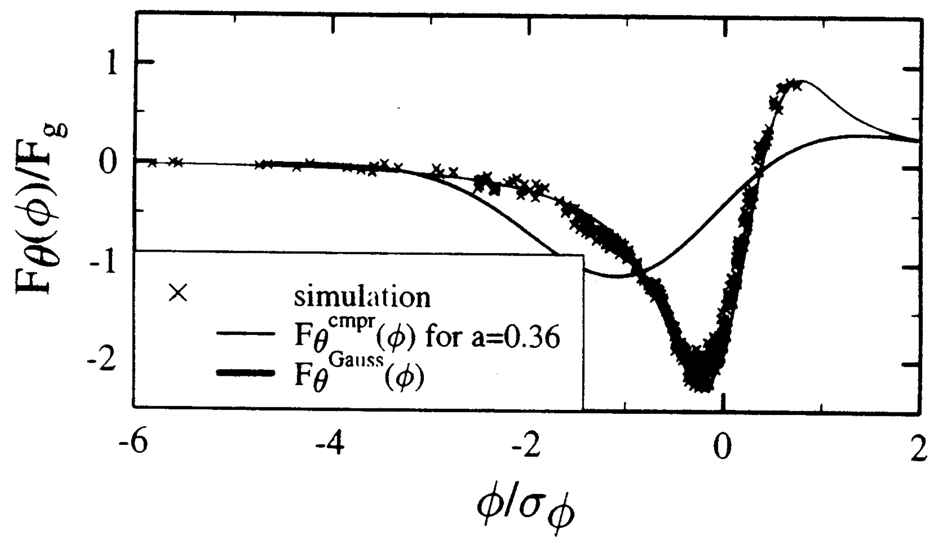
with $$f(y;a) = \int_{-\infty}^{\infty} G(ax)\,(y-x)\,e^{-(y-x)^2/2}\,dx$$

$$G(y) = H(-y)\,e^{-|y|/\sqrt{2}}\,\left|y\right|^{1/6}\,\Gamma\left(\frac{2}{3}\right)\Psi\left(\frac{2}{3},\frac{7}{6};\frac{|y|}{\sqrt{2}}\right) + H(y)\,\left|y\right|^{1/6}\,\Gamma\left(\frac{1}{2}\right)\Psi\left(\frac{1}{2},\frac{7}{6};\frac{|y|}{\sqrt{2}}\right)$$

where $\Psi(\alpha,\gamma;z)$ is the degenerate hypergeometric function :

$$\Psi(\alpha,\gamma;z) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} e^{-zt}\,t^{\alpha-1}\,(1+t)^{\gamma-\alpha-1}\,dt$$

Analytical/Numerical Resutls of Longitudinal CSR Force

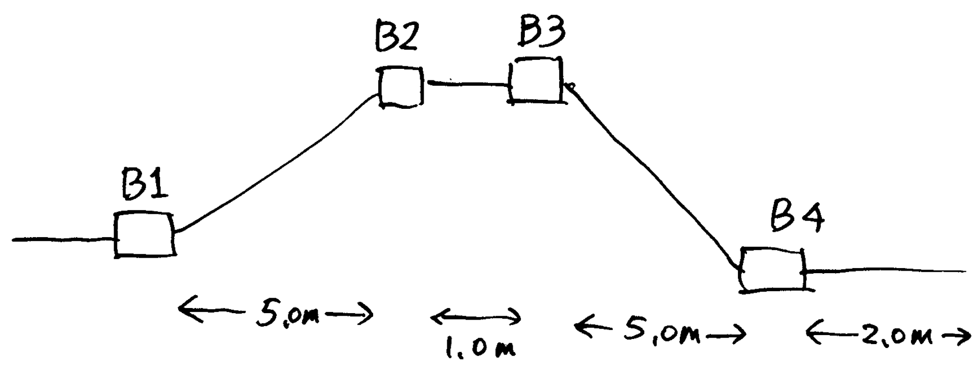


4. Benchmarking Results for the 4-Bend Chicane

Observations

- Transverse size effect shown in Es when x-s has a small slope in a bend
- Or when $\left(\frac{\sigma_s}{R}\right) \approx \left(\frac{\sigma_\perp}{R}\right)^{3/2}$
- $\Delta\delta$ vs. s has less transverse effect if all Es, Ex and Bz are included (agree with cancellation effect)
- With Fr included, the final emittance growth is smaller than if only Es is conssidered.
- Lower energy case has bigger emittance growth
- Uniform beam has less emittance growth than Gaussian beam.

case for Benchmarking : unshielded 4-Bend chicane



Bend magnet Length = 0.5m

Bending angle = 2.77deg

Electron Bunch : Gaussian / Uniform

$E = 5.0\,GeV,\ 0.5\,GeV$

$Q = 1\,nC$

$\sigma_\delta = 0.72\%$

$\sigma_s = 200\,\mu m \rightarrow 20\,\mu m$

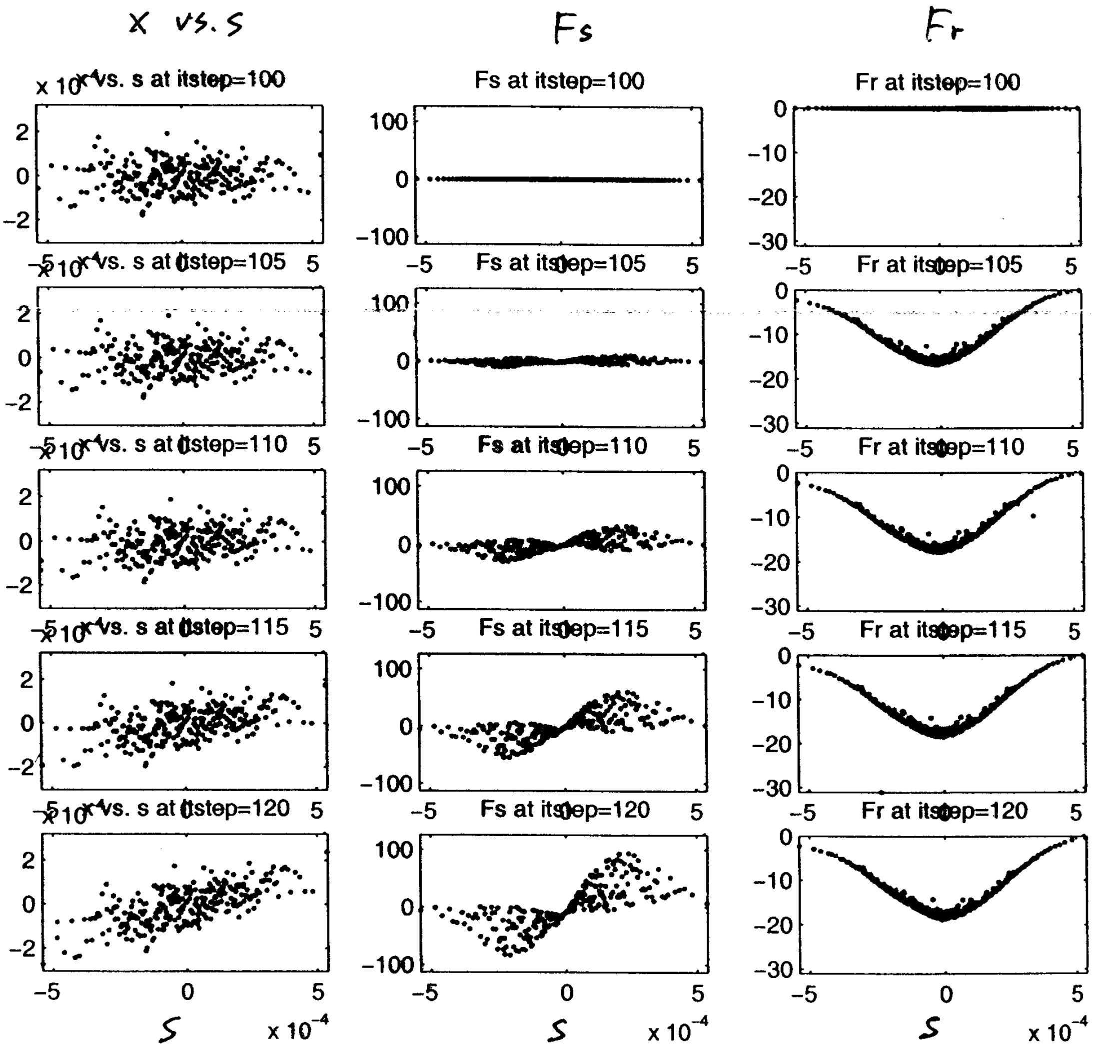
$\epsilon_{nx} = 1.0\ mm\text{-}mrad$

$\beta_x = 40\ m$

$\alpha_x = 2.6$

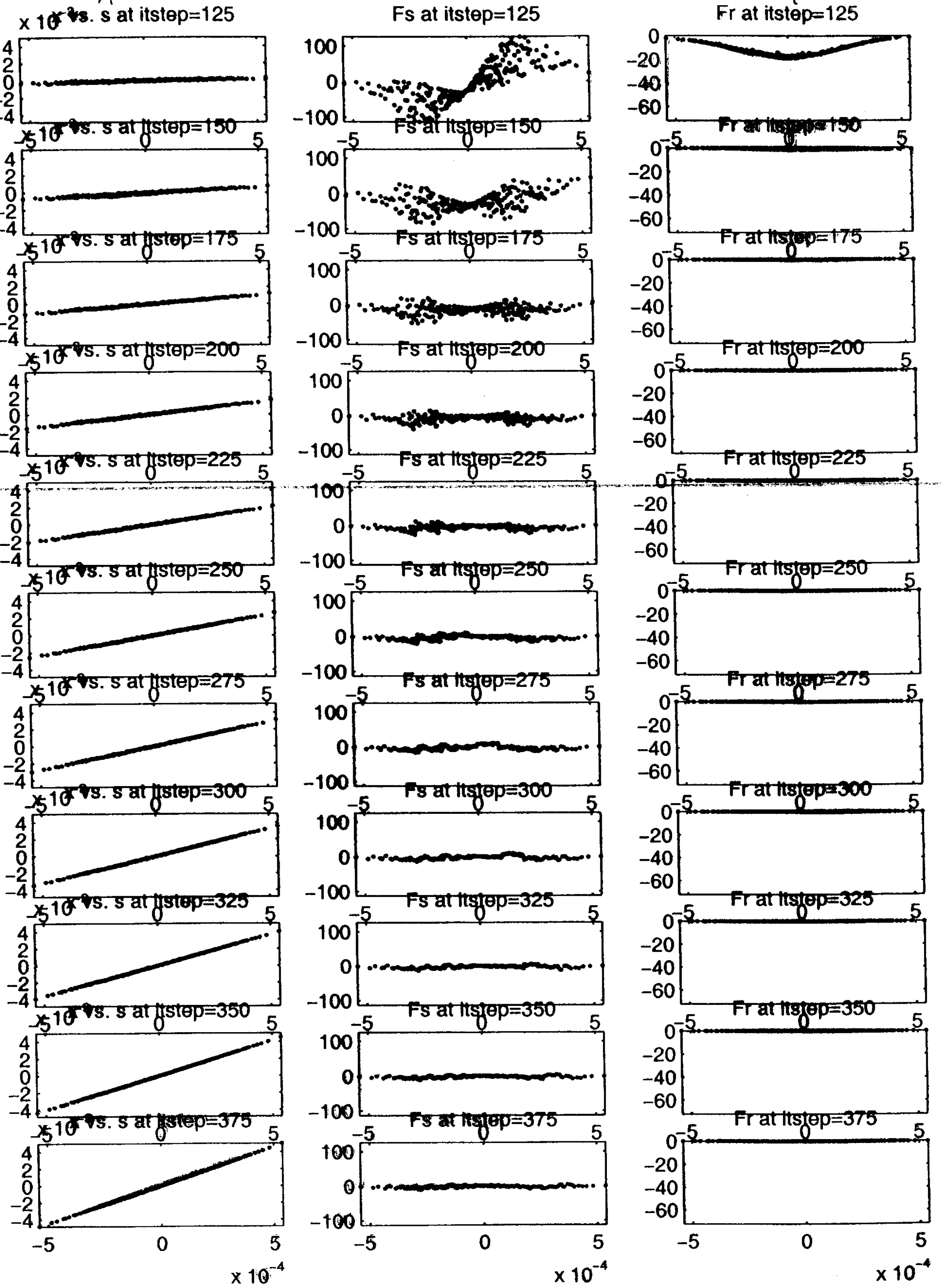
Gaussian , 5 GeV

In 1st Bend

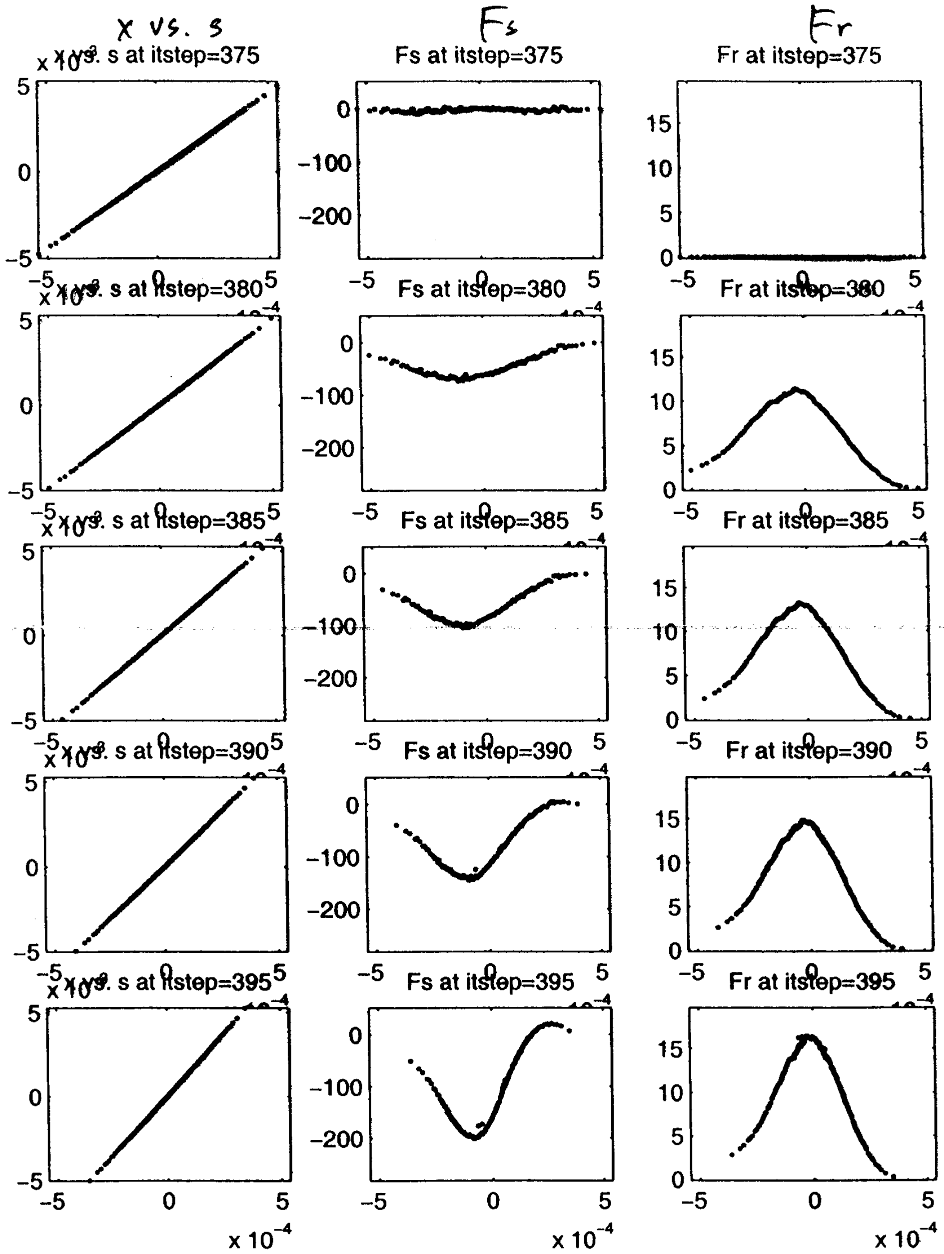


Gaussian In 1st Drift

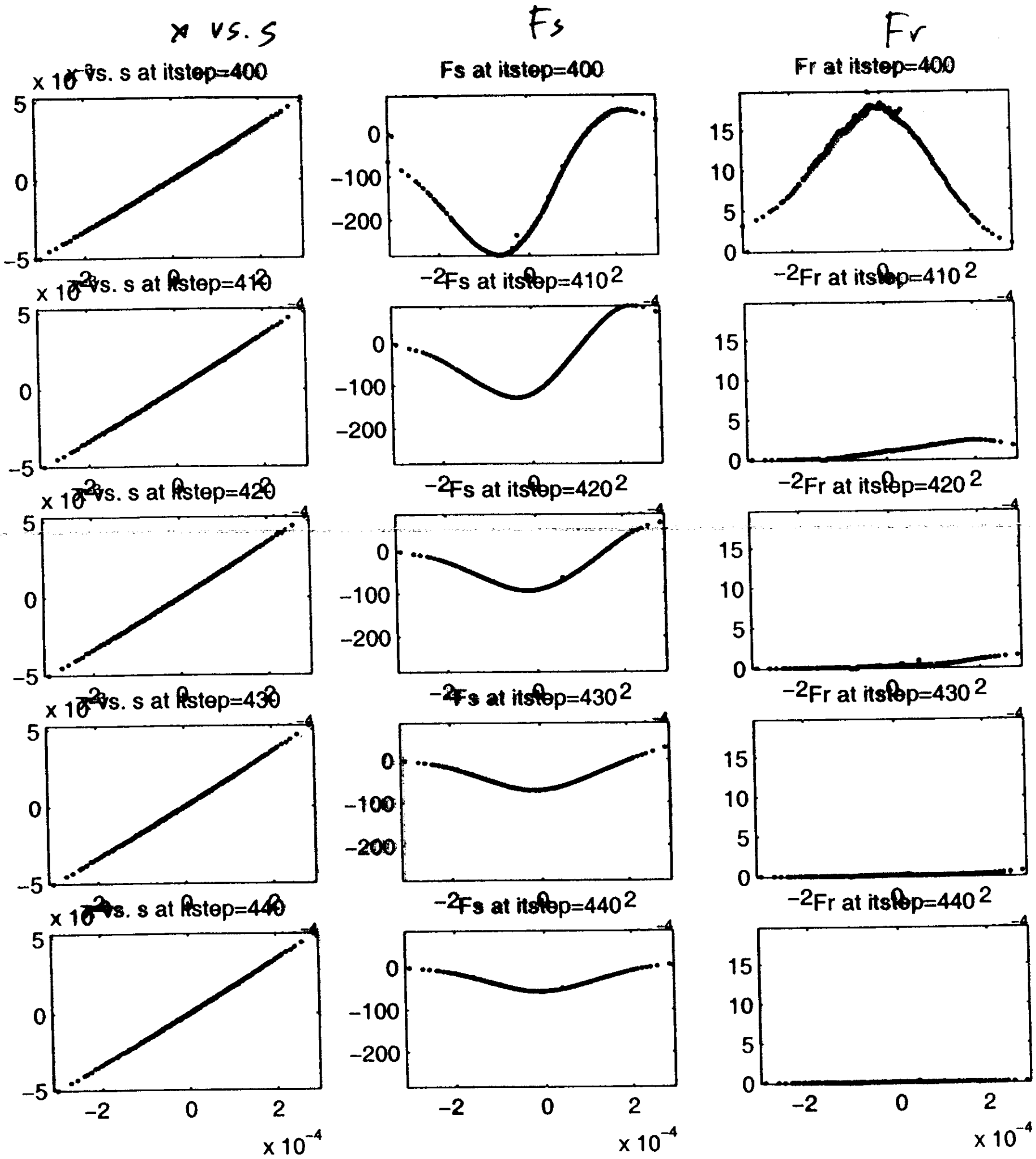
X vs. s Fs Fr



Gaussian In 2nd Bend

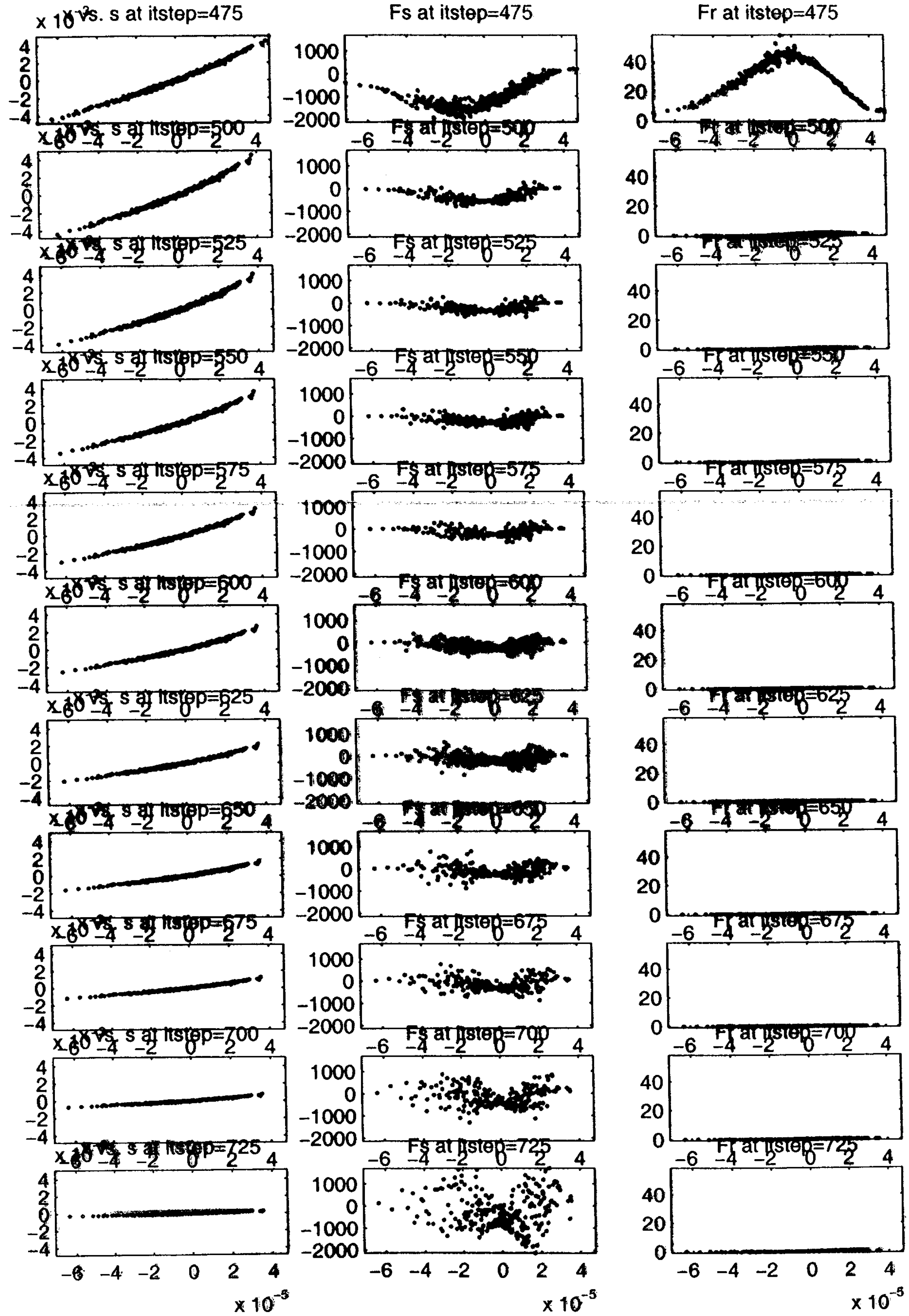


Gaussian, In 2nd Drift

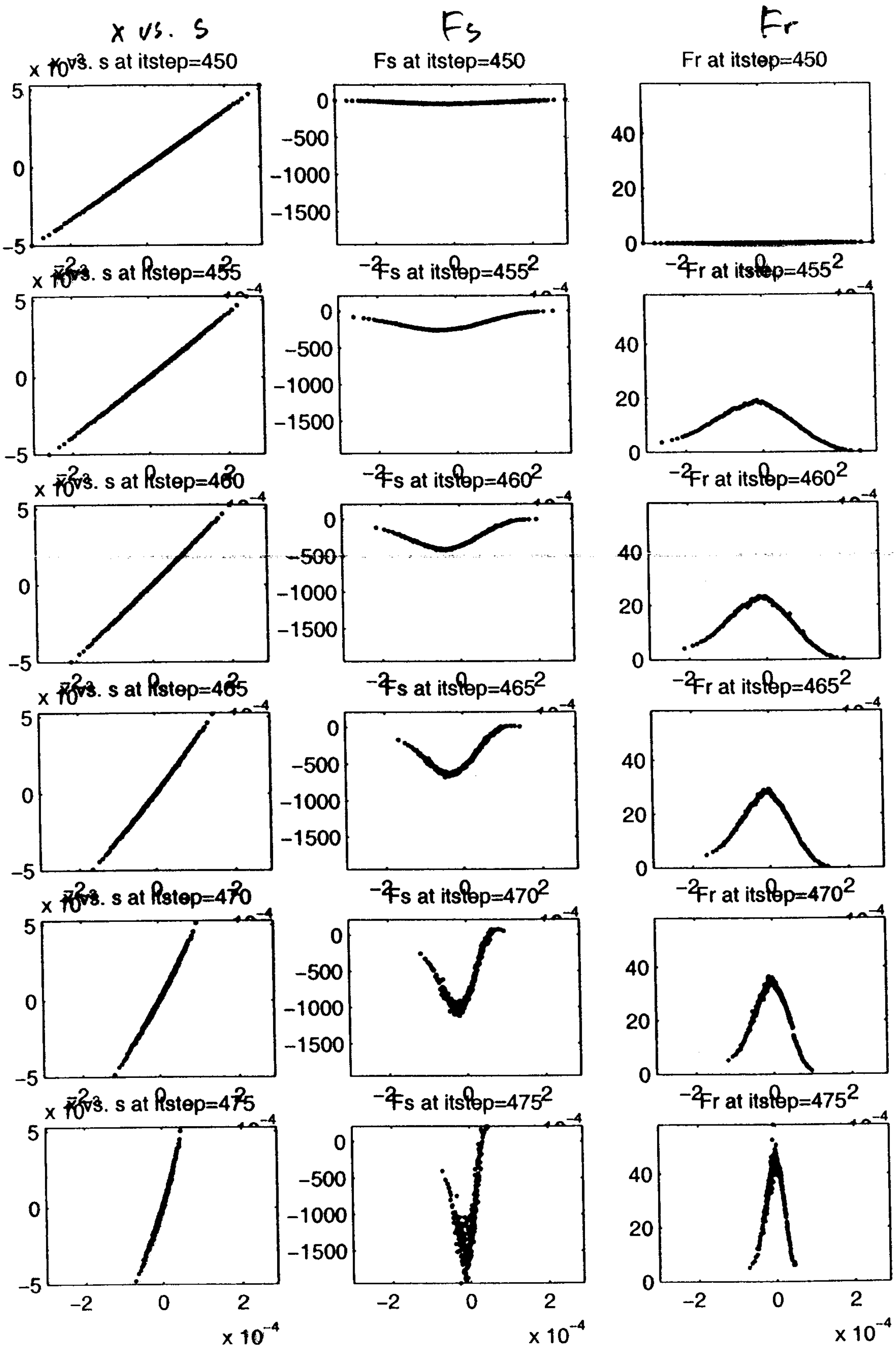


Gaussian In 3rd Drift

X vs. s Fs Fr

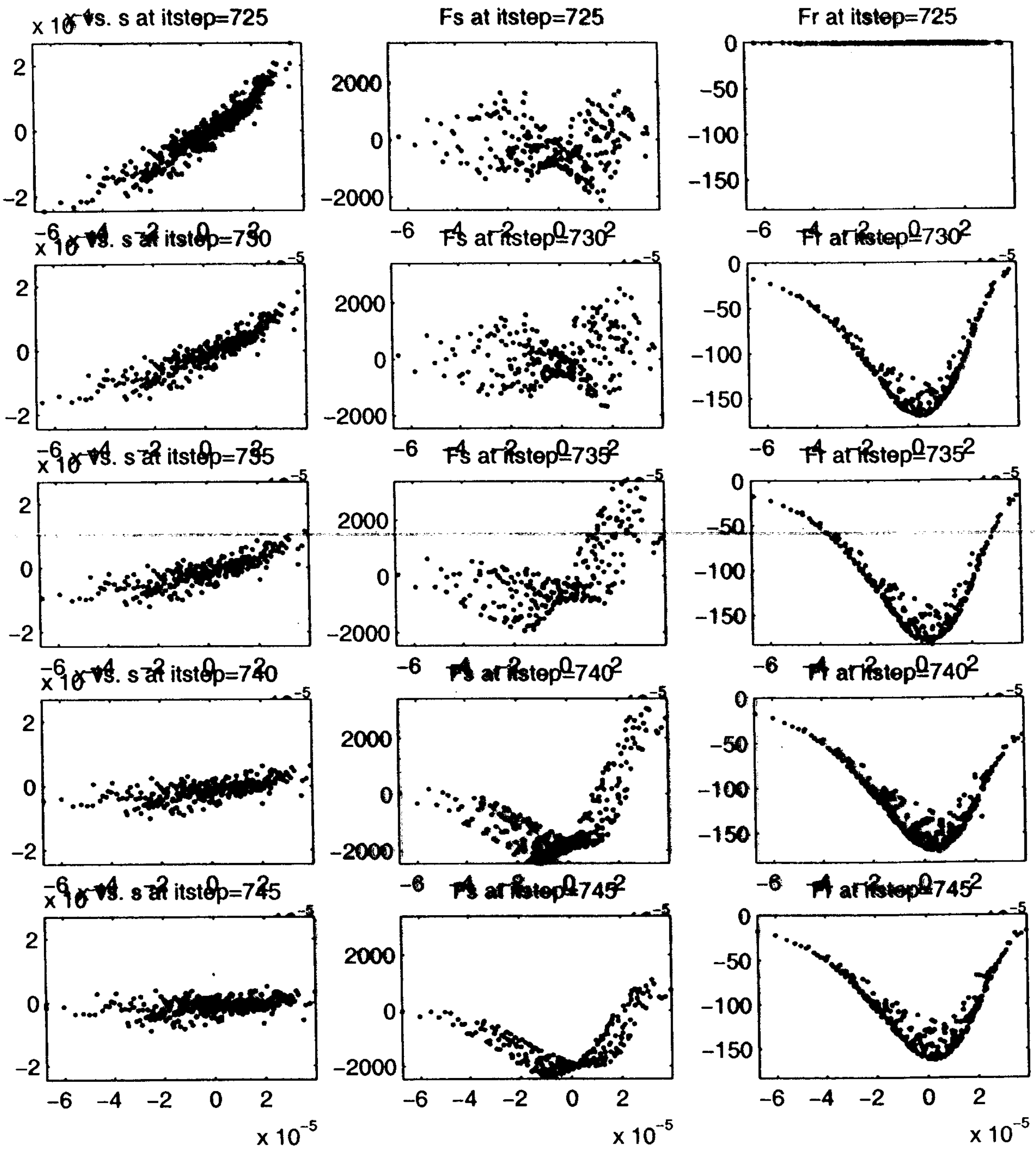


Gaussian, In 3rd Bend

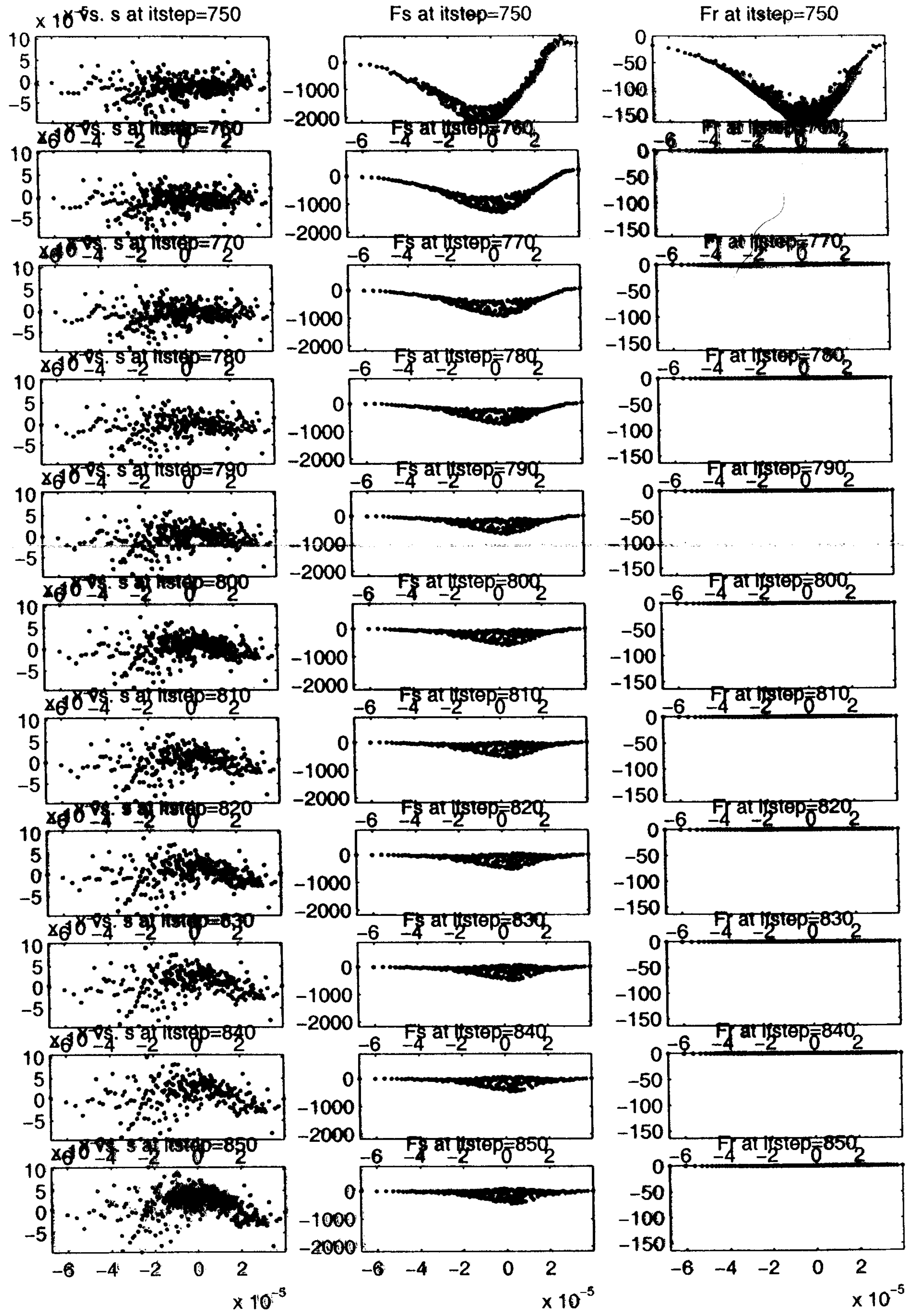


Gaussian In 4th Bend

x vs. s   Fs   Fr

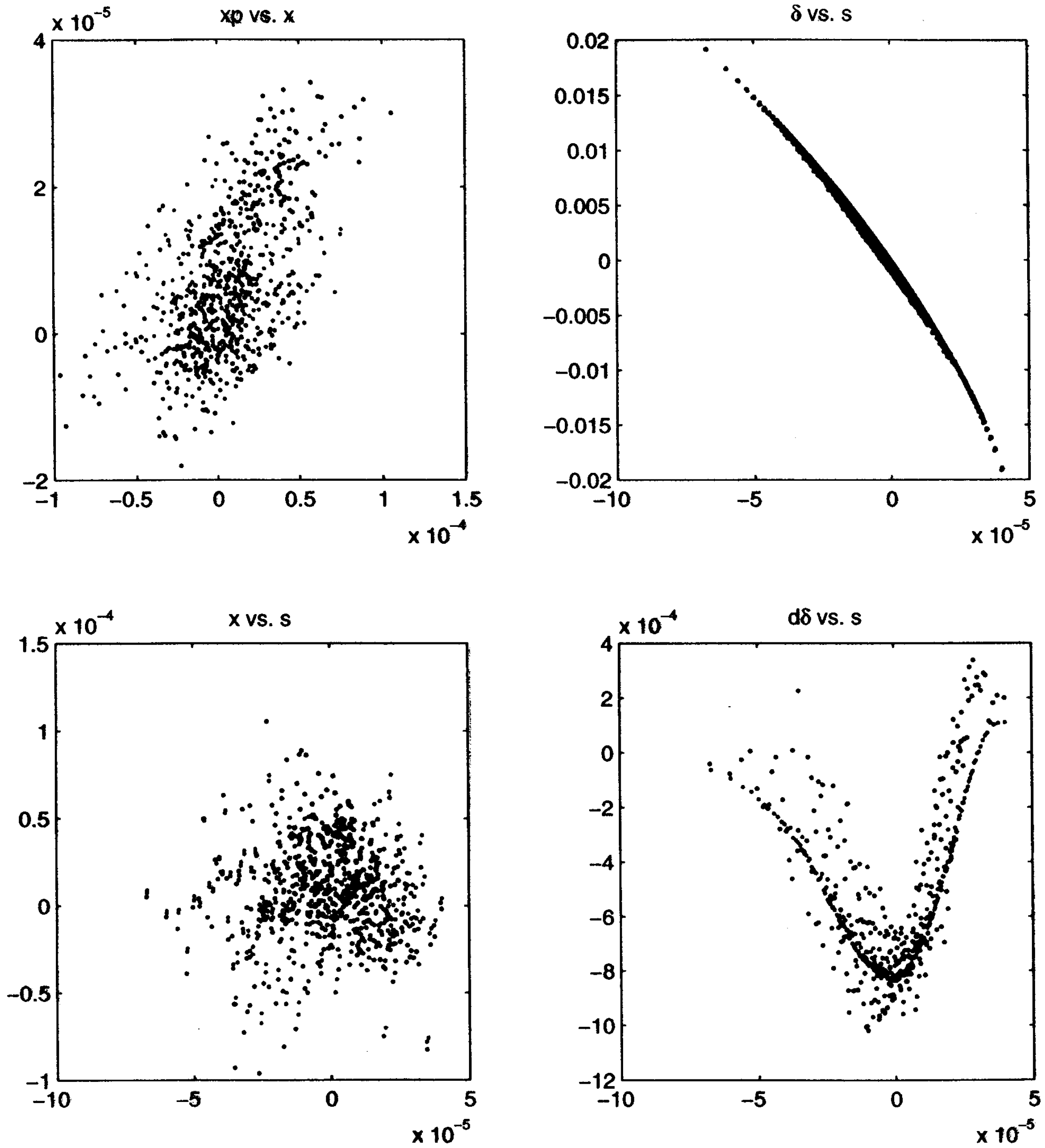


Gaussian In 4th Drift



Gaussian_5GeV_final phase

— no csr
— E_s only
— E_s, E_x, B_z included



Longitudinal CSR Fields

$$E_s = -\frac{\partial \phi}{r\partial \theta} - \frac{\partial A_s}{c\partial t}$$

$$= \frac{1}{\beta_s}\left[-\left(\frac{\partial \phi}{\partial t} + \beta_s \frac{\partial \phi}{r\partial \theta}\right) + \left(\frac{\partial \phi}{\partial t} - \beta_s \frac{\partial A_s}{c\partial t}\right)\right]$$

$$= \frac{1}{\beta_s}\left[-\left[\frac{d\phi}{dt}\right]_s + \frac{\partial}{\partial t}(\phi - \beta_s A_s)\right]$$

line charge in steady state: non inertial space-charge

depend on transverse size.

Energy Conservation

$$\frac{d(\gamma mc^2 + \phi)}{c\,dt} = e\,\frac{\partial(\phi - \vec{\beta}\cdot\vec{A})}{c\,\partial t}$$

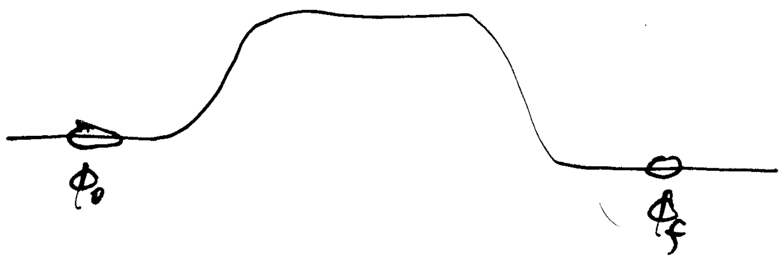
$$(\gamma mc^2)_{t_f} - (\gamma mc^2)_{t_0} = -(\phi_f - \phi_0) + e\int_0^t \frac{\partial(\phi - \vec{\beta}\cdot\vec{A})}{\partial t}\,dt$$

$$= e\int (\beta_s \cdot \vec{E}_s + \beta_r \cdot \vec{E}_r)\,dt\,c$$

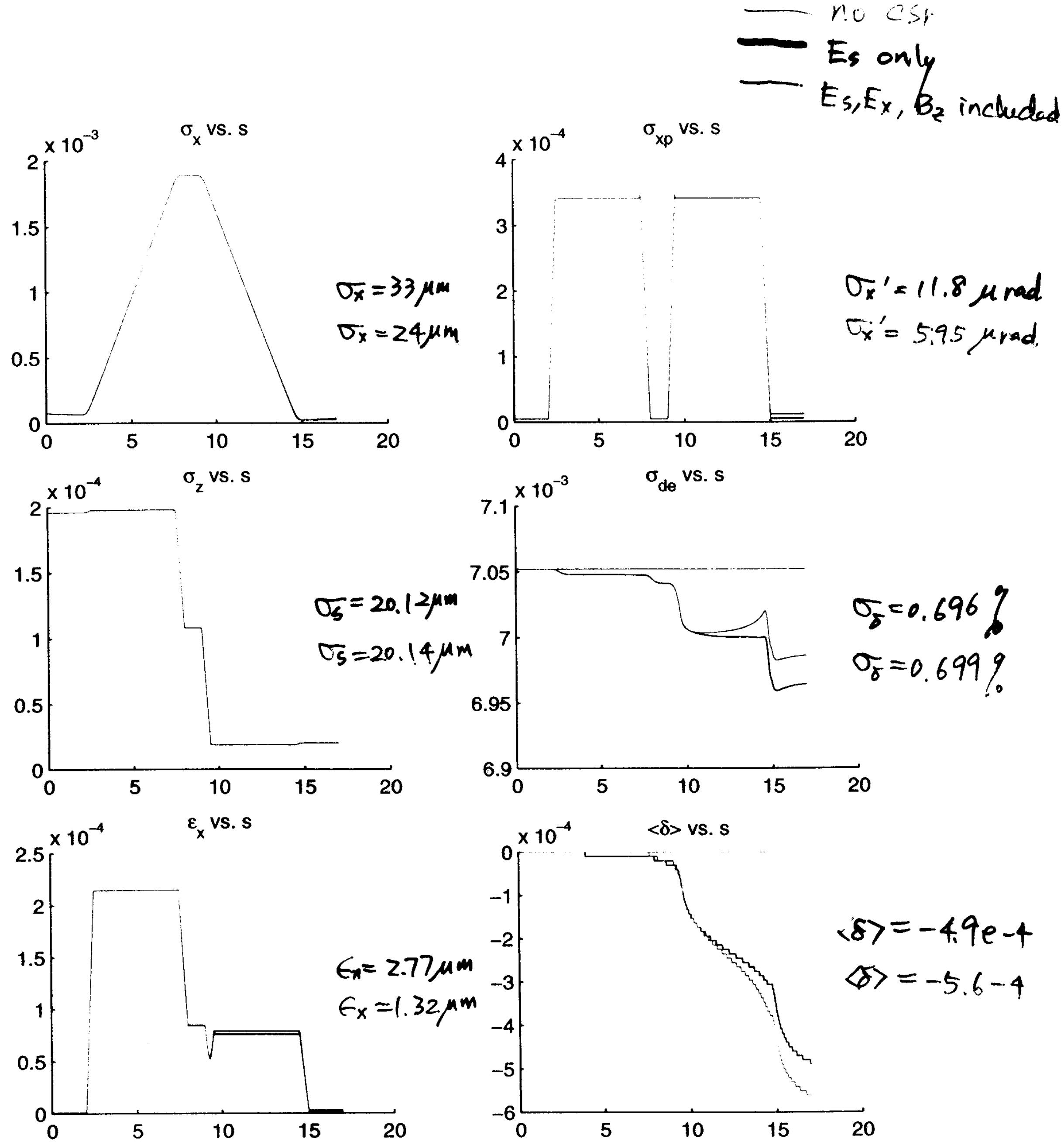
$\downarrow$

$2\sin^2\frac{\Delta\theta}{2}$

$$\frac{\partial}{\partial t}\int \frac{(1-\cos\Delta\theta)\,n(\theta', x')}{|\vec{r} - \vec{r}'|}\,d\Delta\theta\,dr'$$



Gaussian _ 5GeV _ rmsplt



3. Effects of Centrifugal / Noninertial Space Charge Force

- Equation of Motion

$$\frac{d(\gamma m \mathbf{v})}{dt} = e(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}) = e\left[-\nabla(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}) - \frac{d\mathbf{A}}{cdt}\right]$$

- Radial Component

$$\frac{d(\gamma\beta_r)}{cdt} - \beta_\theta\left(\frac{\gamma\beta_\theta}{r} - \frac{\gamma\beta_\theta}{\rho}\right) = \frac{e}{mc^2}\left[-\frac{\partial}{\partial r}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}) + \beta_\theta\frac{A_\theta}{r} - \frac{dA_r}{cdt}\right]$$

- First Order of Eq. for $x = r - \rho$:

$$\frac{d^2x}{c^2dt^2} + \frac{x}{\rho^2} = \frac{\Delta\gamma}{\gamma_0\rho} + \frac{e}{\gamma_0 mc^2}\left[-\frac{\partial}{\partial r}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}) + \beta_\theta\frac{A_\theta}{r} - \frac{dA_r}{cdt}\right]$$

$$= \frac{e}{\gamma_0 mc^2}\left[\frac{1}{\rho}\int_0^t\underbrace{\left(-\frac{d\Phi}{dt} + \frac{\partial}{\partial t}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A})\right)dt}_{\boxed{\text{noninertial space charge force}}} + \underbrace{\left(-\frac{\partial}{\partial r}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}) + \beta_\theta\frac{A_\theta}{r} - \frac{dA_r}{cdt}\right)}_{\boxed{\text{centrifugal space charge force}}}\right]$$

$-(\Phi - \Phi_0)$

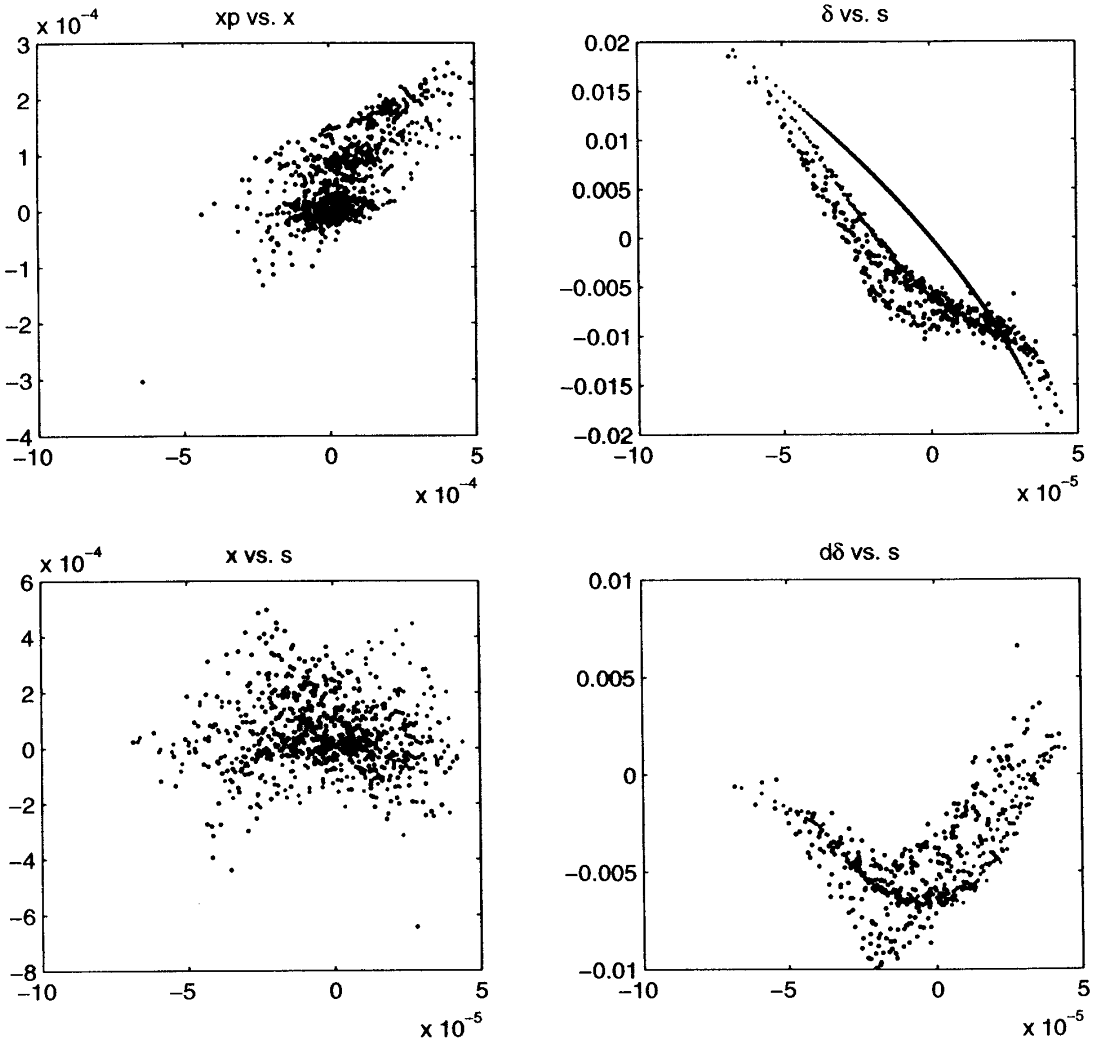
$$= \frac{e}{\gamma_0 mc^2}\left[\frac{\Phi_0}{\rho} + \frac{1}{\rho}\int_0^t\underbrace{\frac{\partial}{\partial t}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A})\,dt}_{\text{longitudinal CSR force}} - \underbrace{\frac{\partial}{\partial r}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A})}_{\text{centripetal space charge force}} + \beta_\theta\underbrace{\frac{A_\theta - \Phi}{\rho}}_{\text{residual}} - \underbrace{\frac{dA_r}{cdt}}_{\boxed{\text{negligible}}}\right]$$

Gaussian_ 0.5GeV_ final phase.

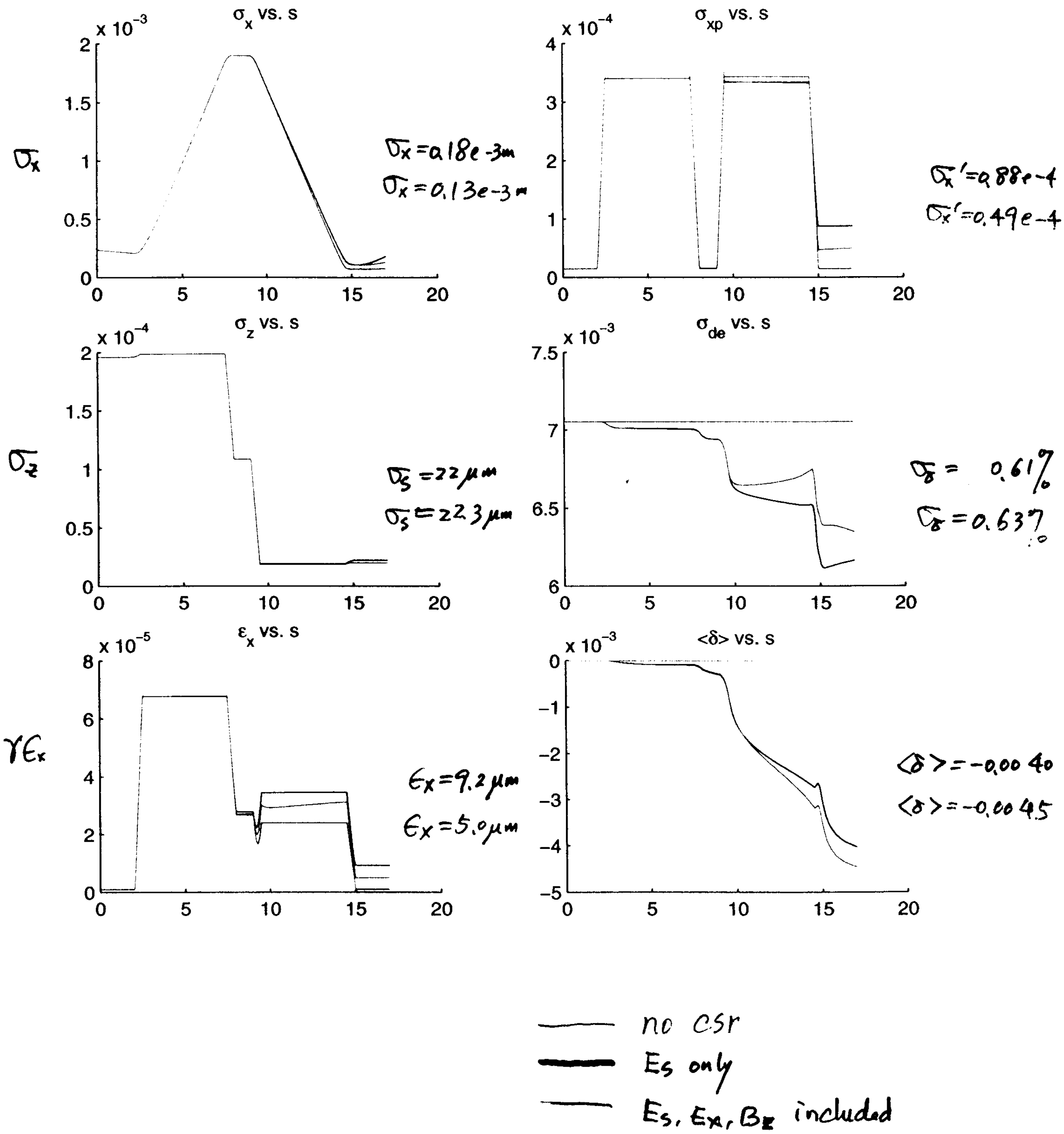
— no csr

— E_s only

— E_s, E_x, G_s



Gaussian_0.5GeV_rmsplt

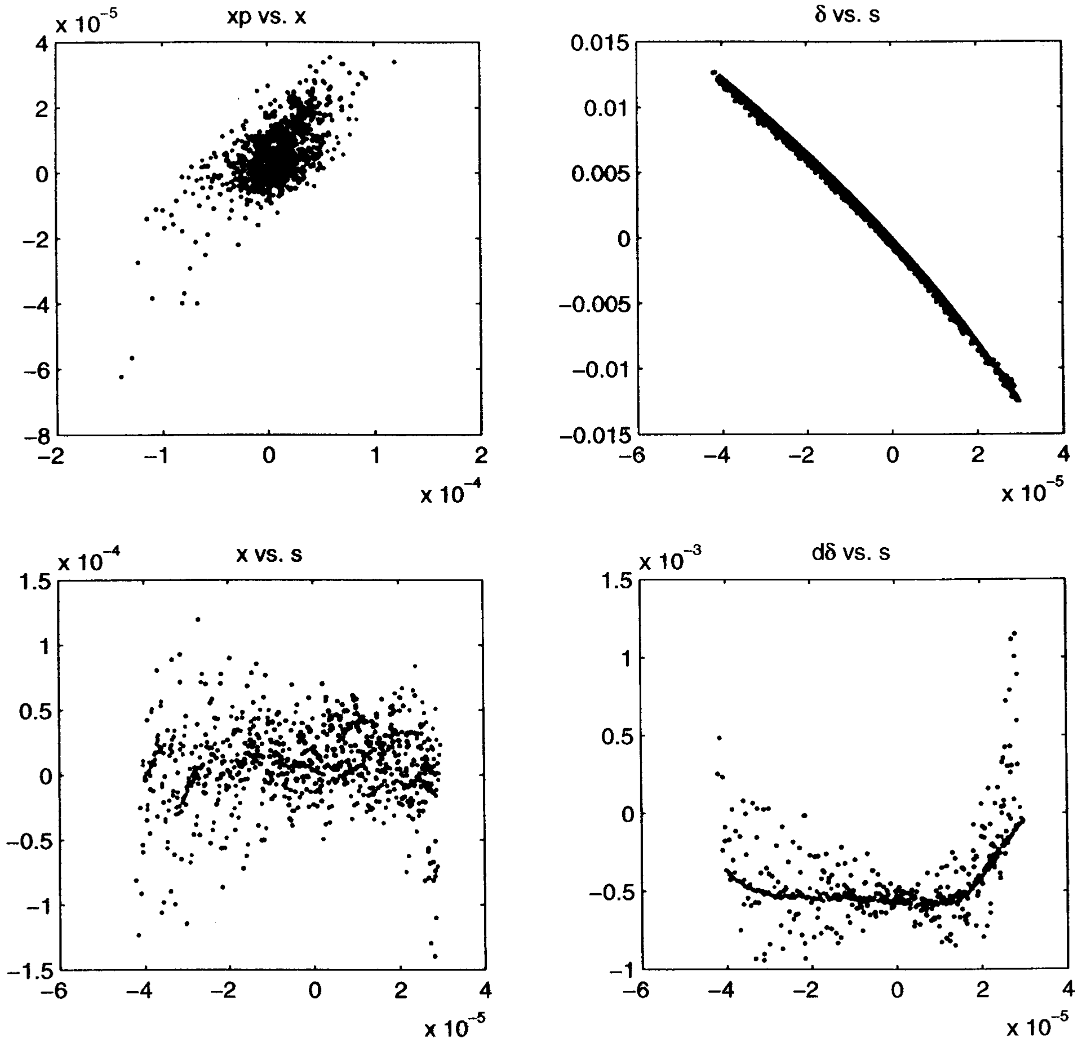


Uniform-5GeV-finalphase

— no csr

— E_s only

— E_s, E_x, B_z included

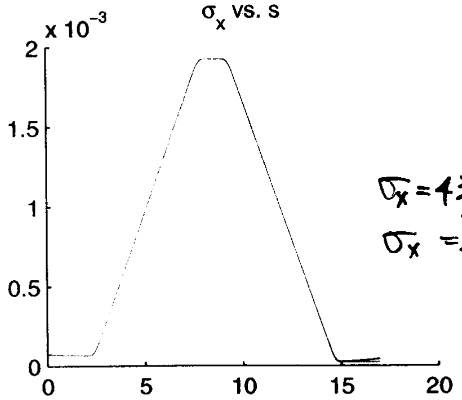


Uniform - 5GeV - rmsplt

· no csr

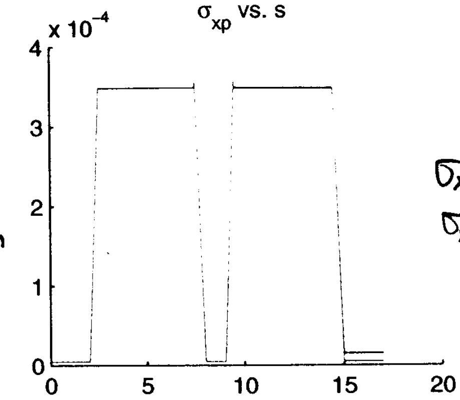
Es only

Es, Ex, Bz included



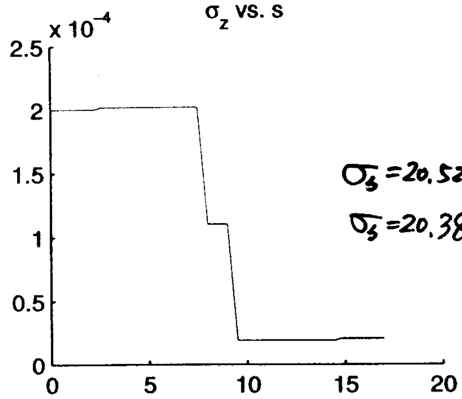
$\sigma_x = 43 \mu m$

$\sigma_x = 24 \mu m$



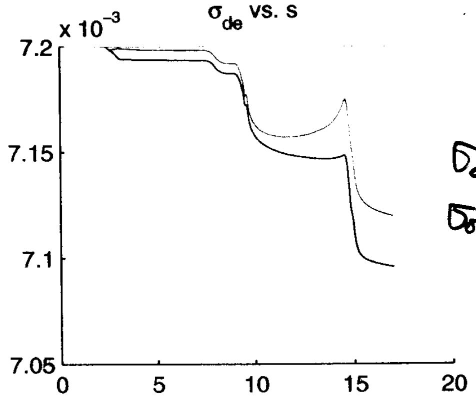
$\sigma_x' = 14 \mu rad$

$\sigma_x' = 4.9 \mu rad$



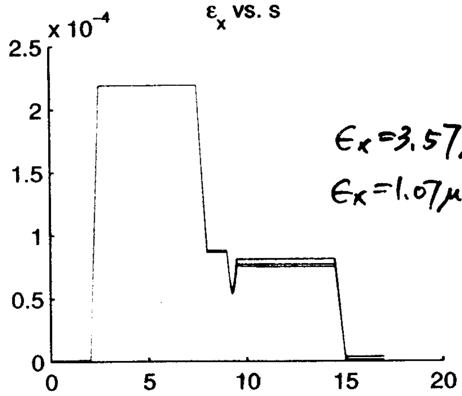
$\sigma_z = 20.52 \mu m$

$\sigma_z = 20.38 \mu m$



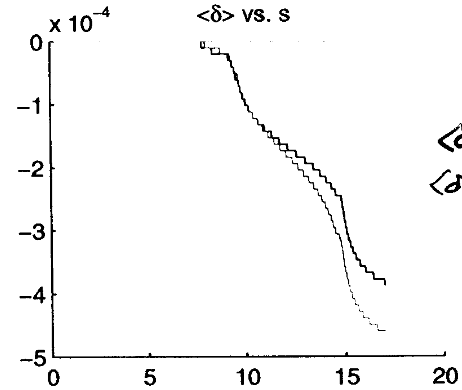
$\sigma_\delta = 0.7096\%$

$\sigma_\delta = 0.712\%$



$\epsilon_x = 3.57 \mu m$

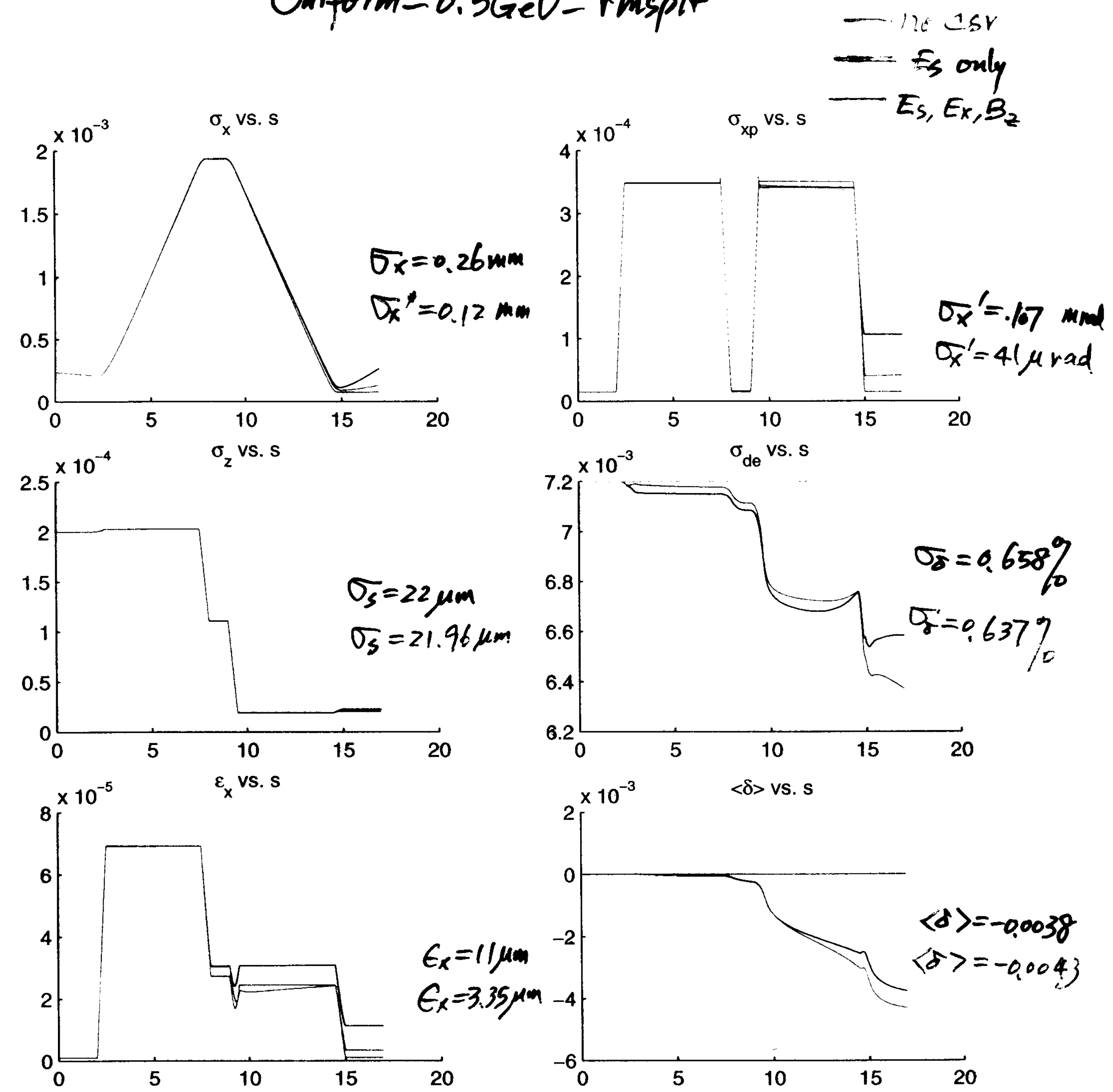
$\epsilon_x = 1.07 \mu m$



$\langle\delta\rangle = -3.88e-4$

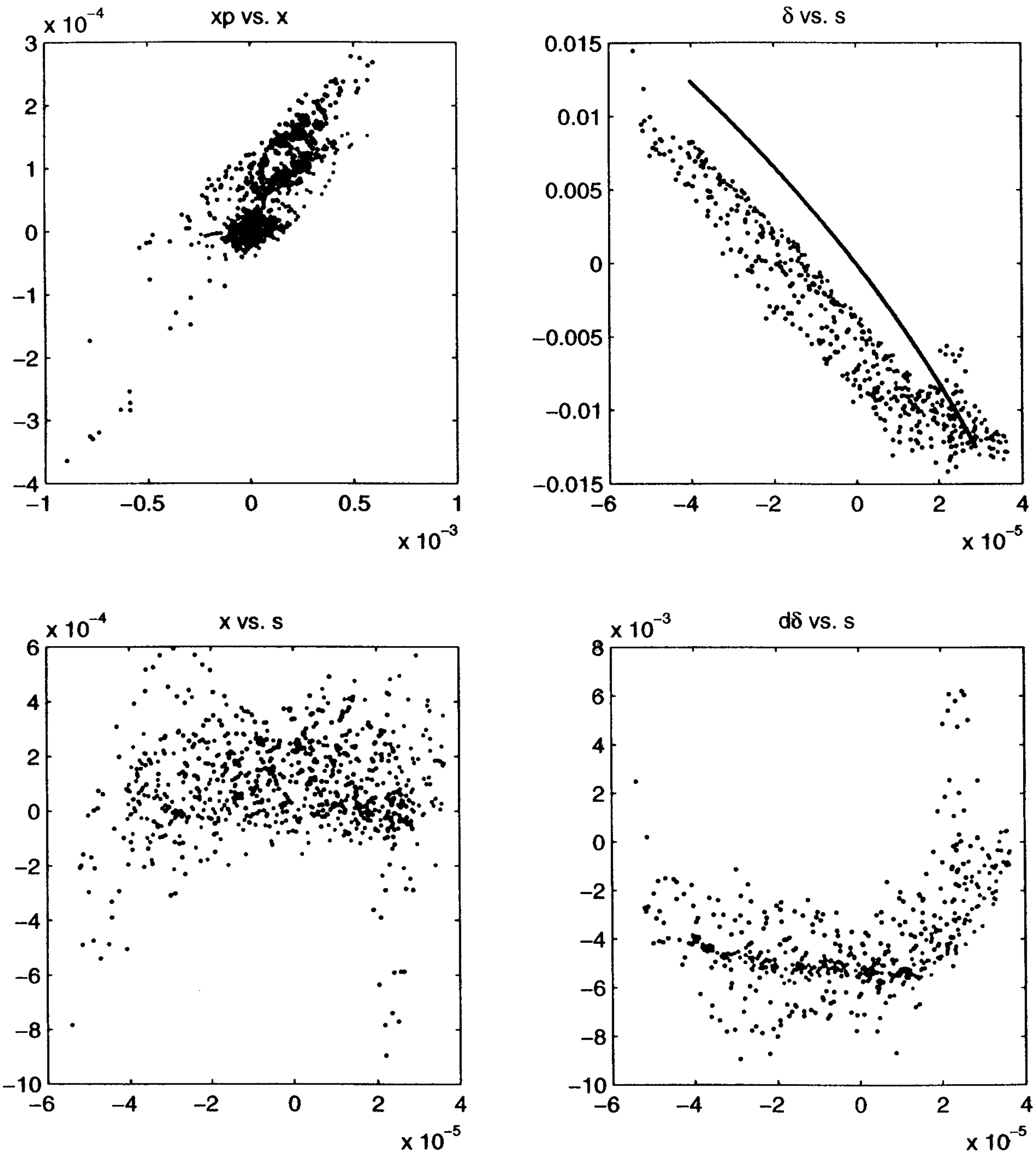
$\langle\delta\rangle = -4.6e-4$

Uniform-0.5GeV-rmsplt



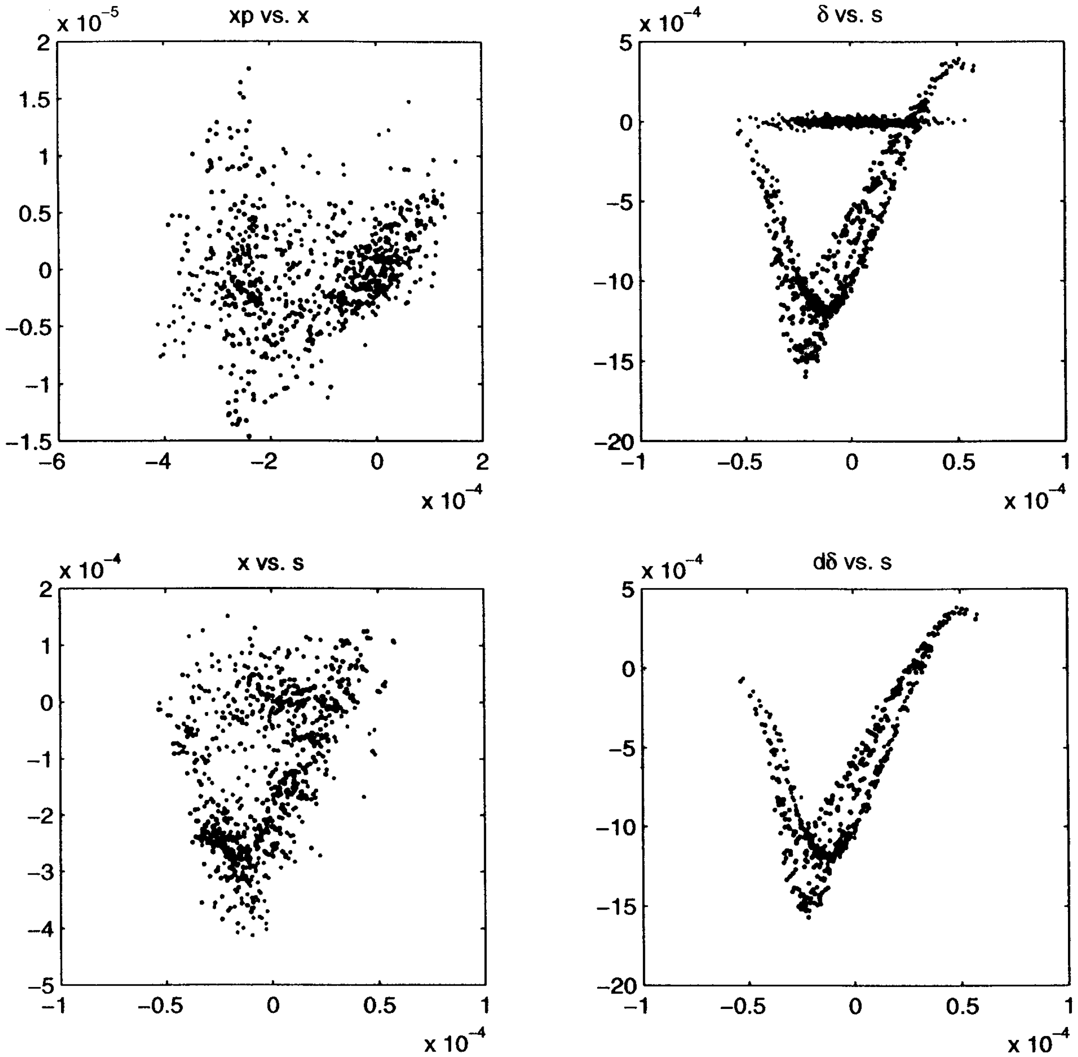
Uniform - 0.5 GeV - ~~no~~ final phase

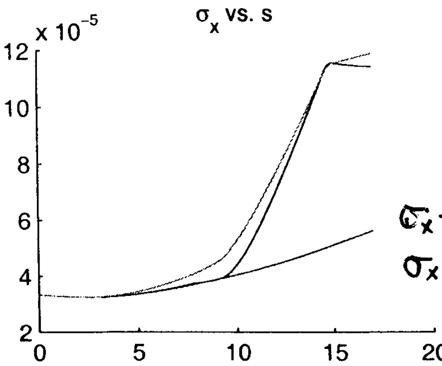
no CSR
Es only
Es, Ex Bz includ



No Compression – final phase

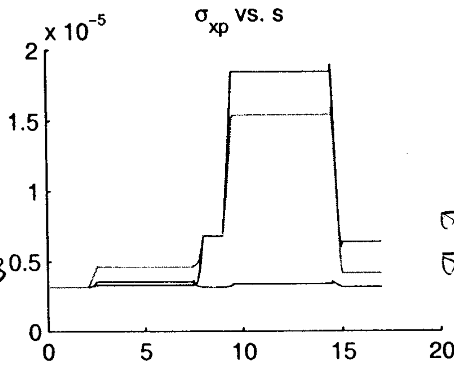
no CSR
Es only
Es, Ex, Bz included





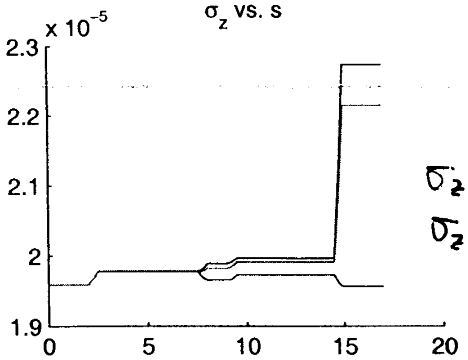
$\sigma_x = .11e\text{-}3$

$\sigma_x = .119e\text{-}3$



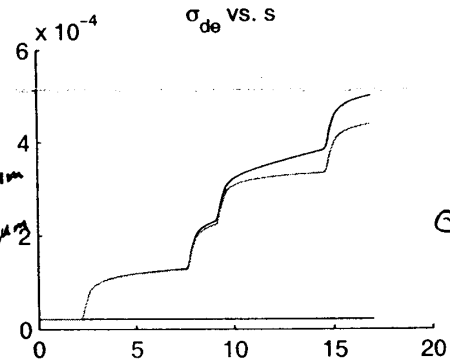
$\sigma_{x'} = .637e\text{-}5$

$\sigma_{x'} = .414e\text{-}5$



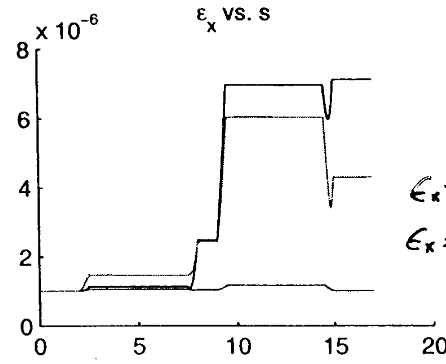
$\sigma_z = 22.7\,\mu m$

$\sigma_z = 22.15\,\mu m$



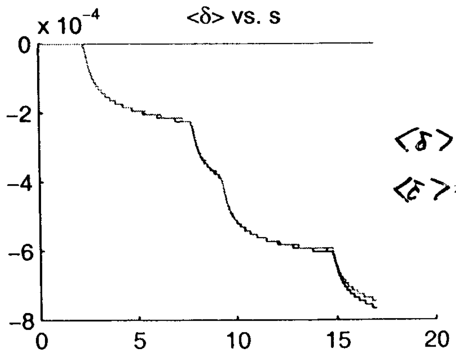
$\sigma_\delta = 0.5e\text{-}3$

$\sigma_\delta = 0.439e\text{-}3$



$\epsilon_x = 7.1\,\mu m$

$\epsilon_x = 4.3\,\mu m$

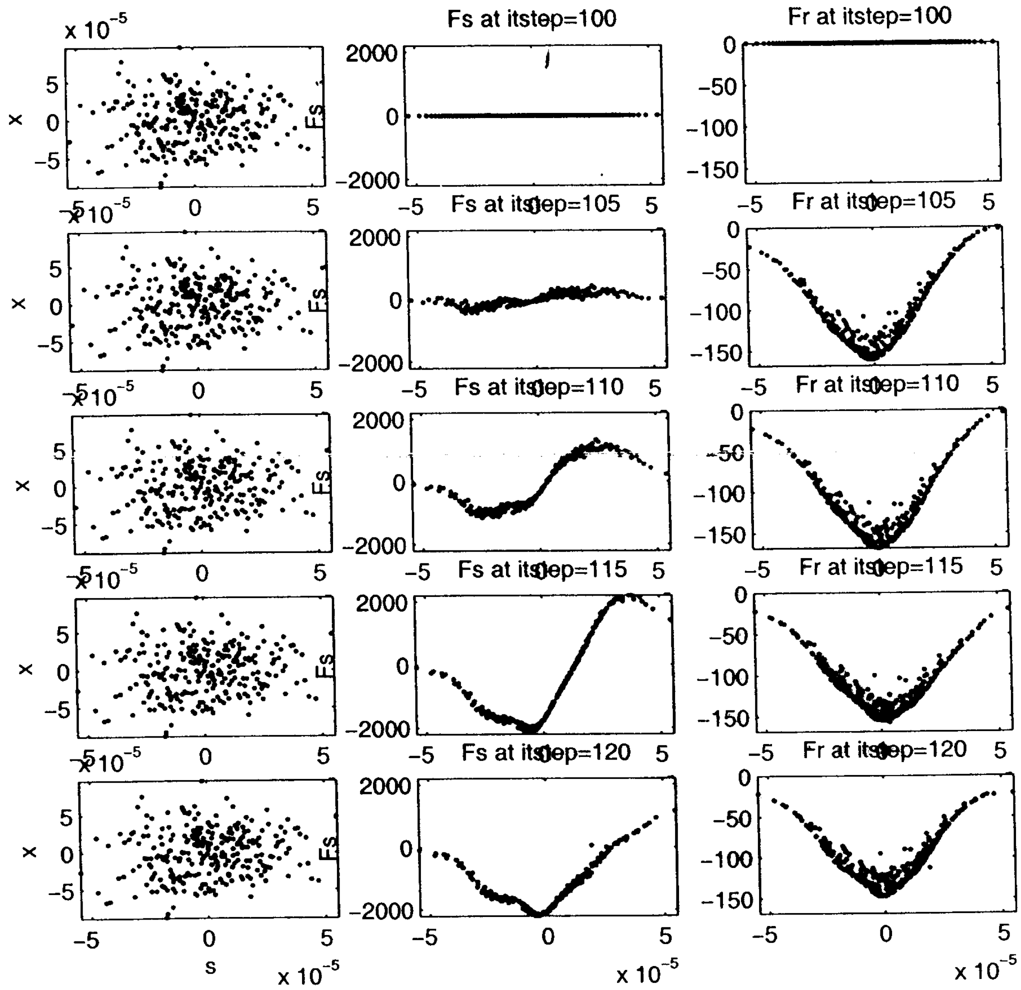


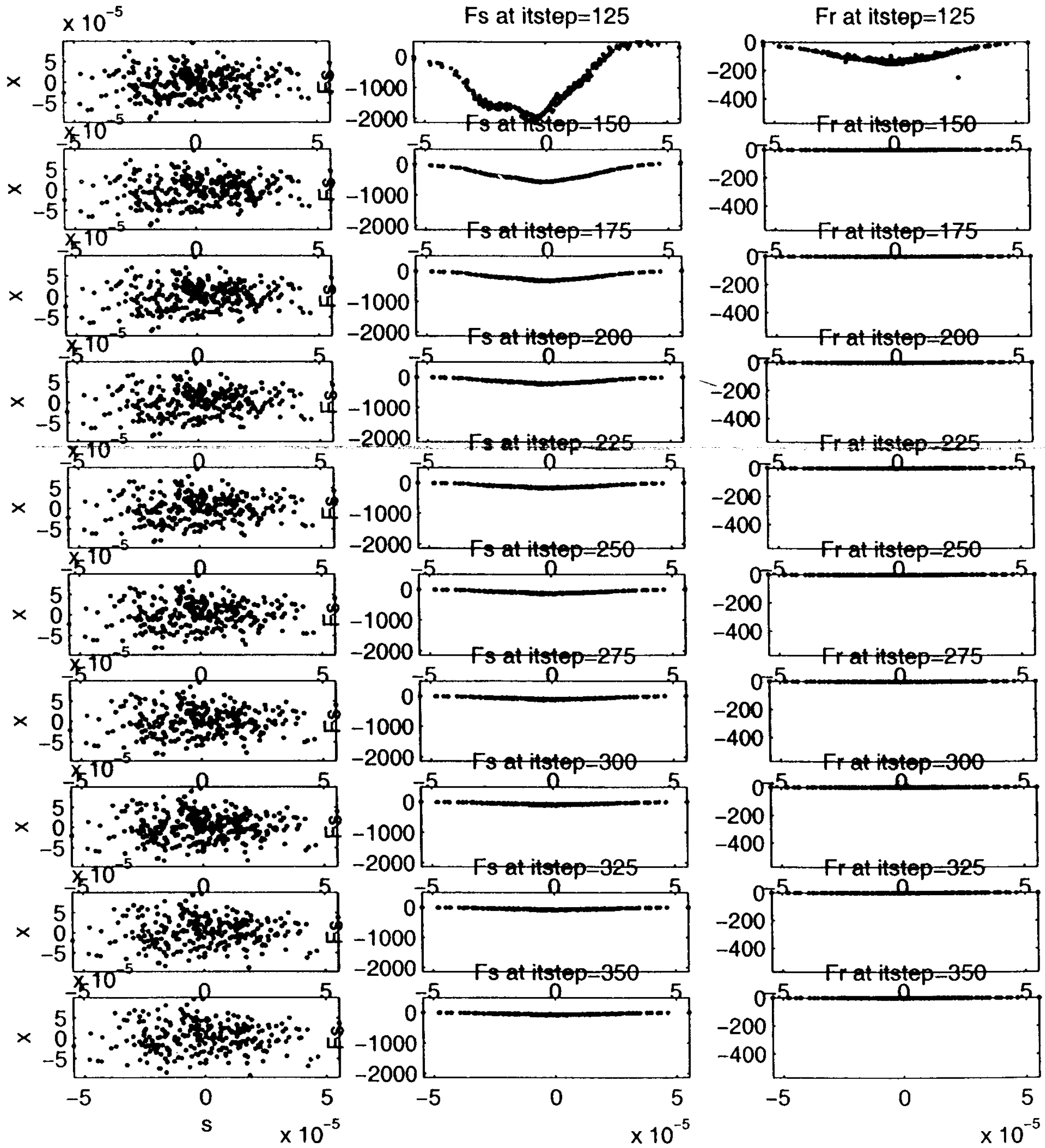
$\langle\delta\rangle = -7.66e\text{-}4$

$\langle\delta\rangle = -7.46e\text{-}4$

Uniform, 1st Bend

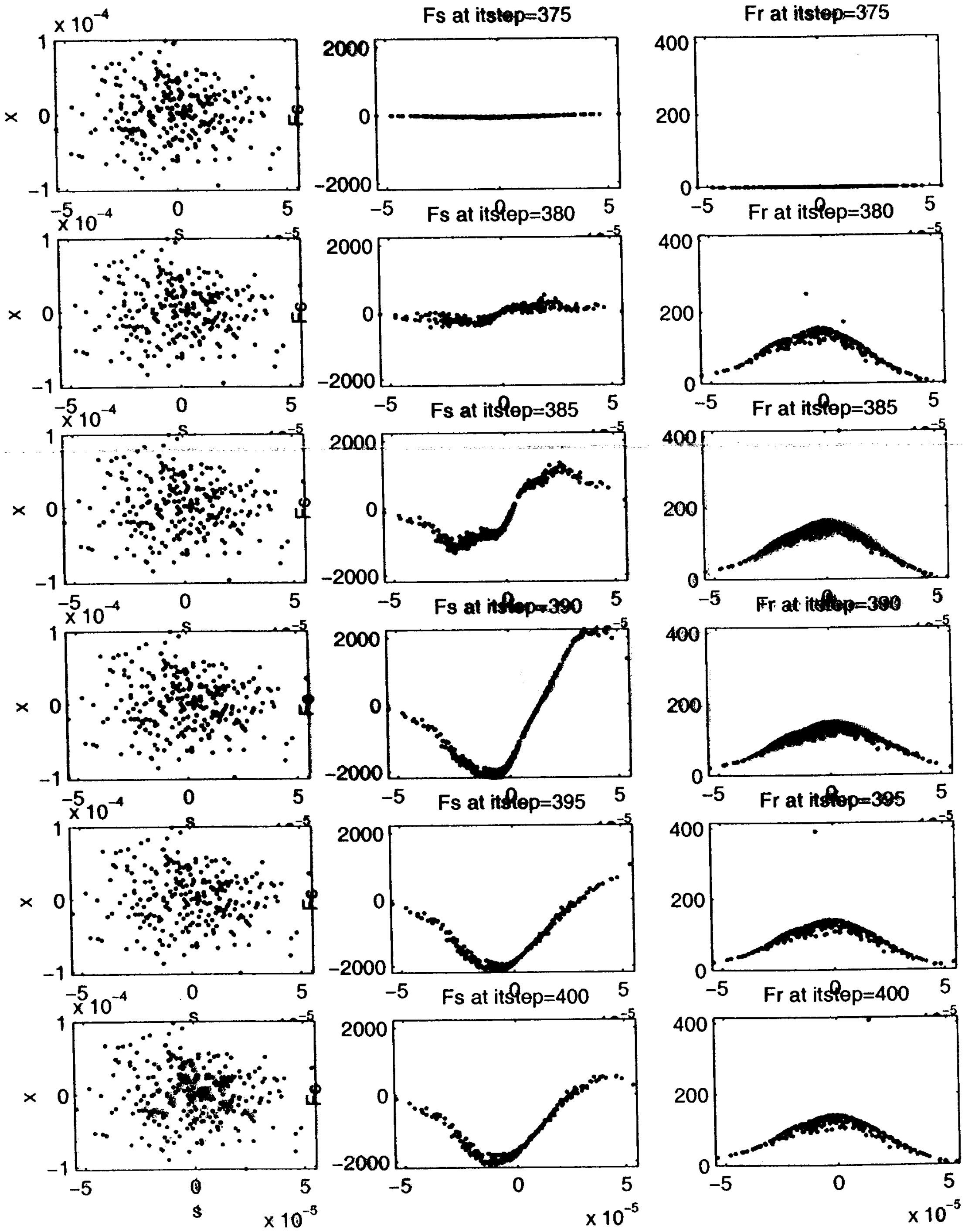
X vs S



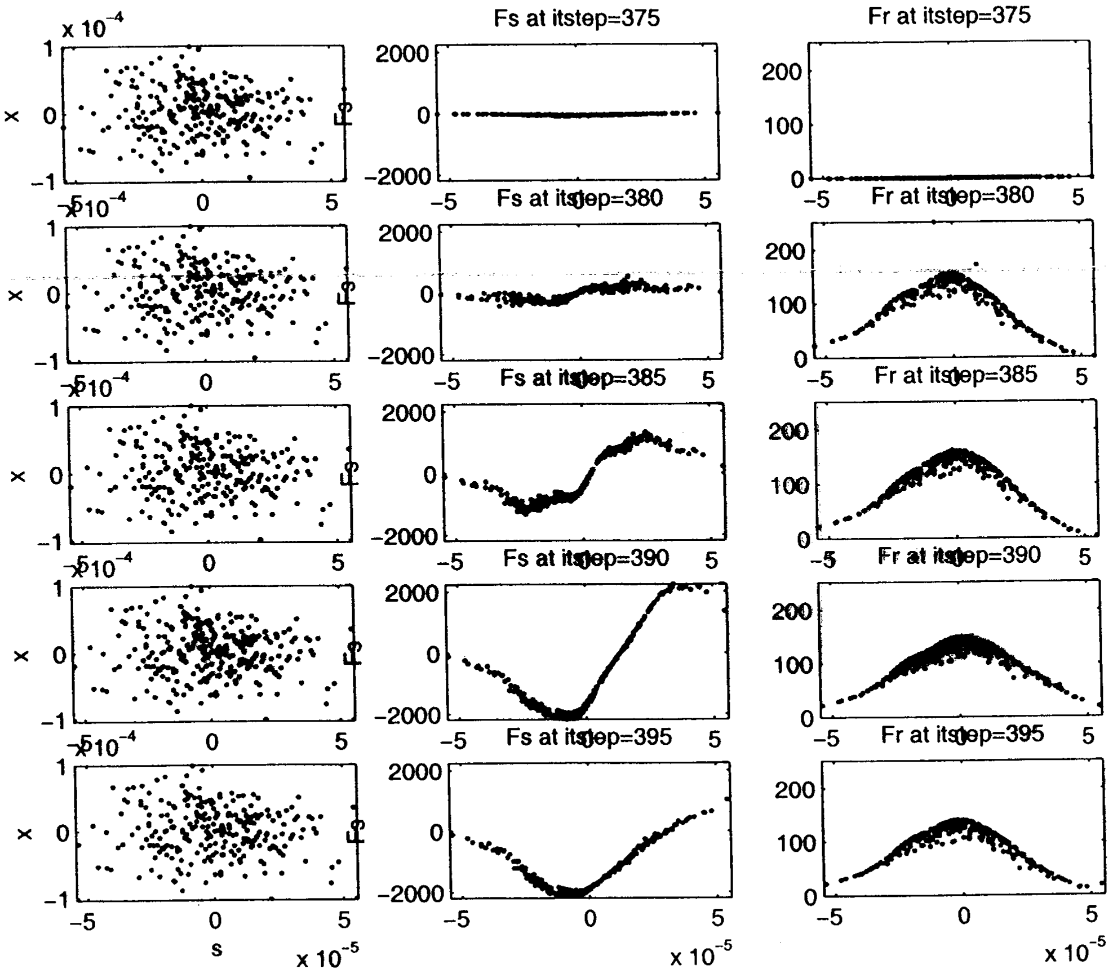


Fs at itstep=125
Fr at itstep=125
Fs at itstep=150
Fr at itstep=150
Fs at itstep=175
Fr at itstep=175
Fs at itstep=200
Fr at itstep=200
Fs at itstep=225
Fr at itstep=225
Fs at itstep=250
Fr at itstep=250
Fs at itstep=275
Fr at itstep=275
Fs at itstep=300
Fr at itstep=300
Fs at itstep=325
Fr at itstep=325
Fs at itstep=350
Fr at itstep=350
x
Fs
s
x 10^-5

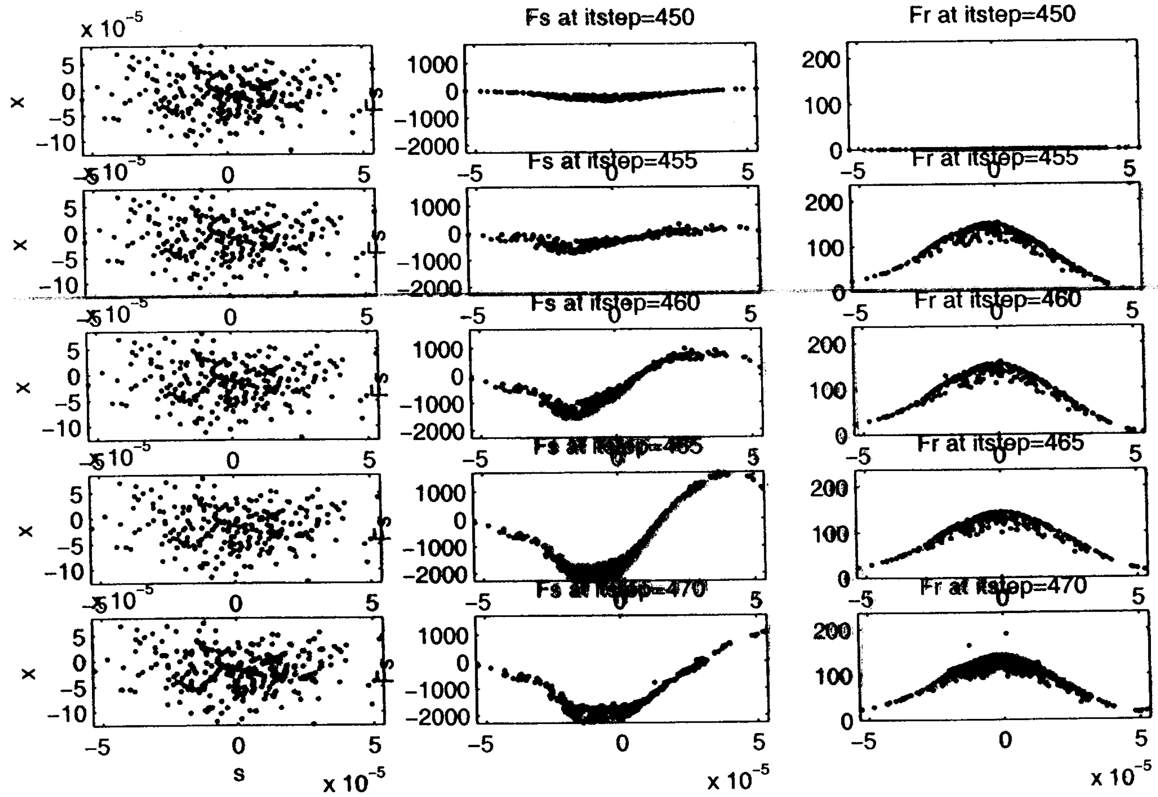
Uniform 2nd Bond



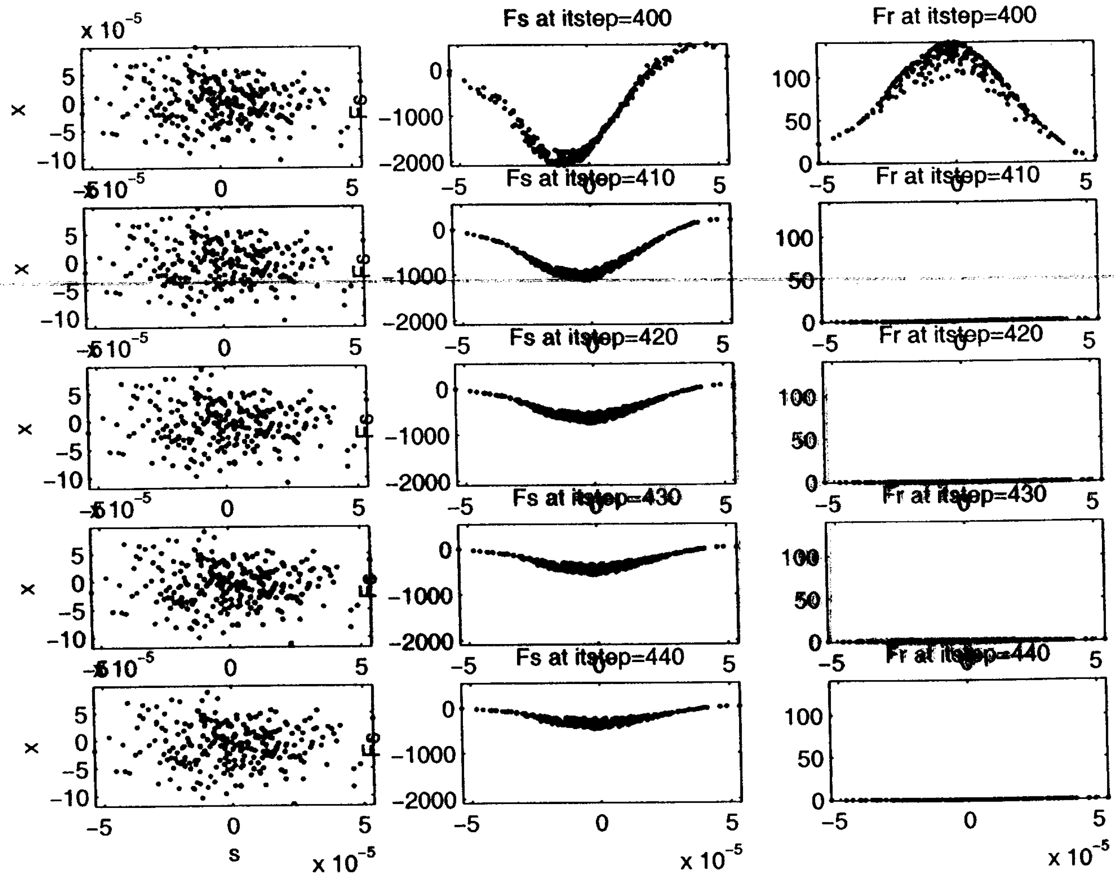
Uniform 2nd drift



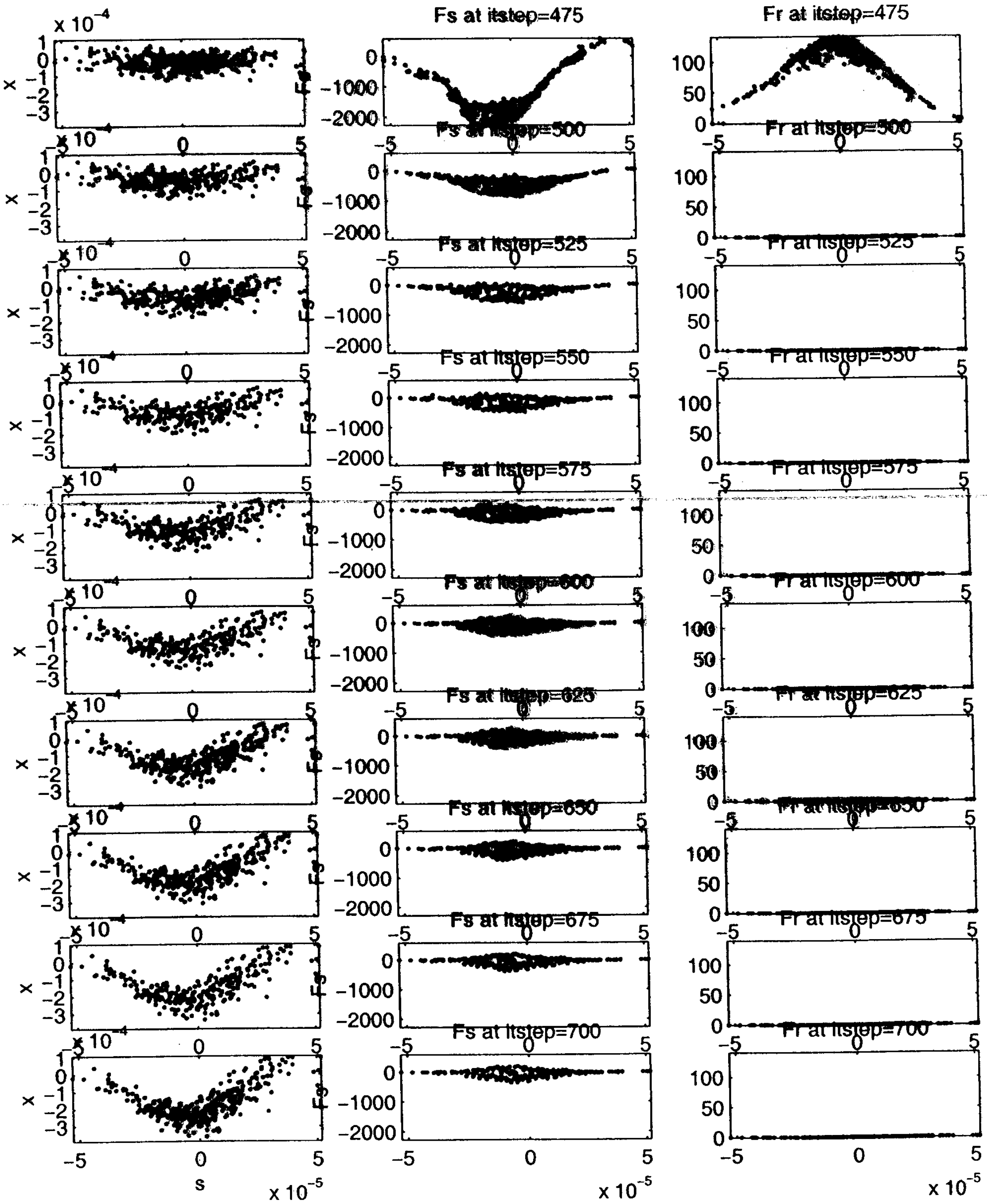
Uniform 3rd Bond



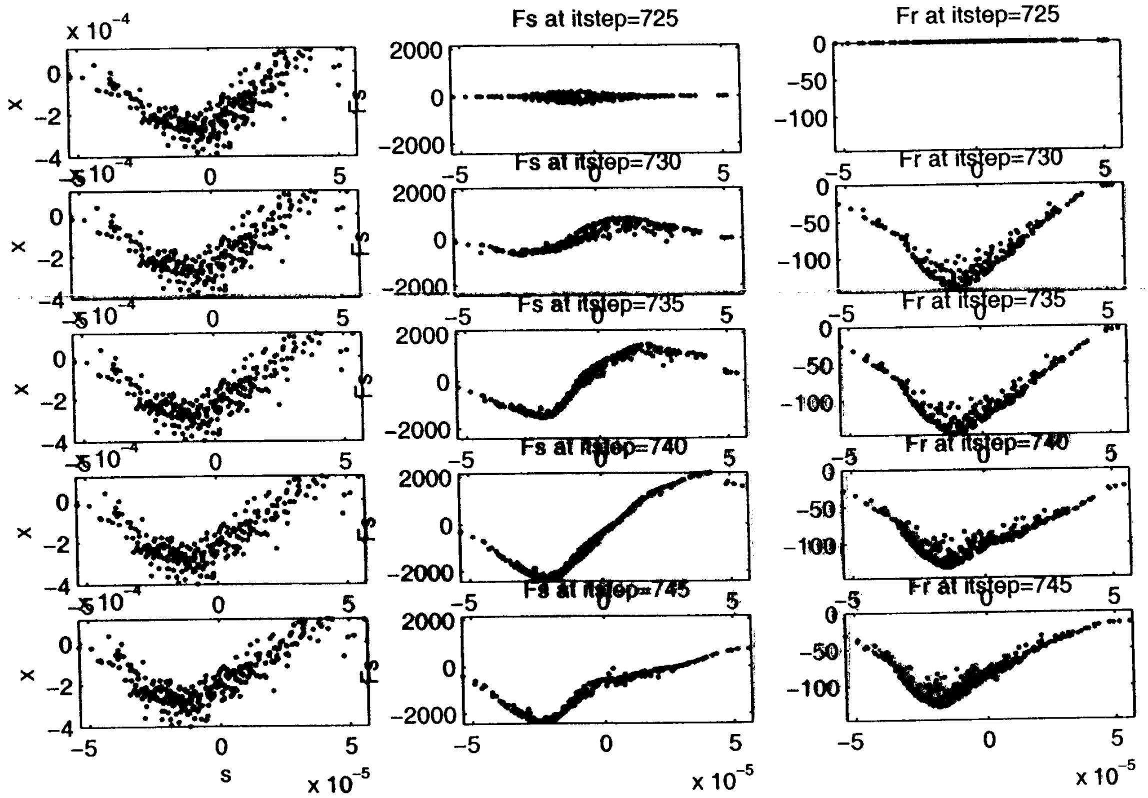
Uniform 3rd Drift



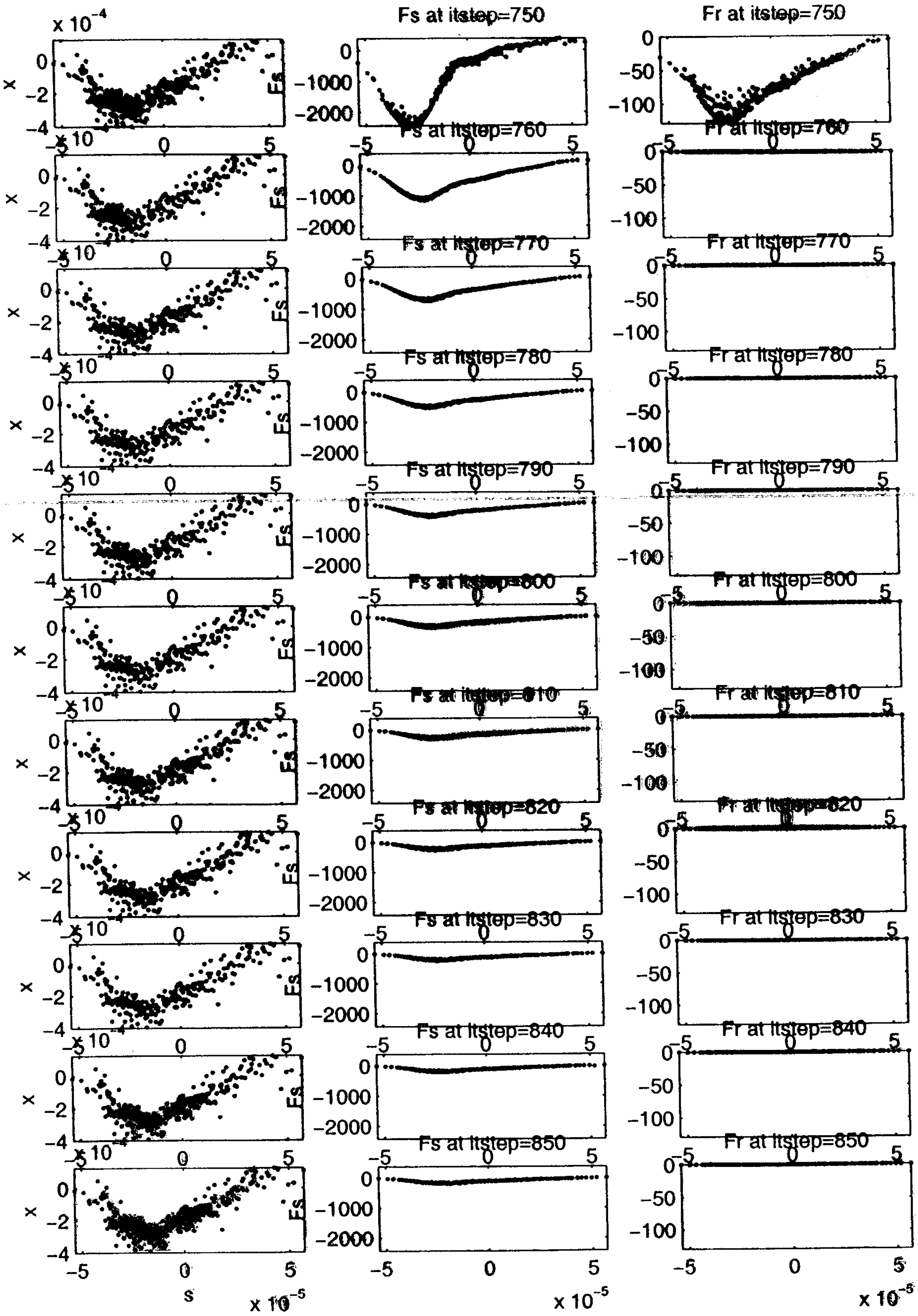
Uniform ~~4th Bend~~ 3rd Drift



Uniform 4th Bend



Uniform 4th draft



5. Conclusion

- A simulation of CSR effects based on macroparticle model

  has been developed.
- The simulation of the 4-Bend Chicane shows the interplay of longitudinal and transverse effect. More analytical works need to be done for further understanding.
- The code is not suitable for study microbunching instability. An ideal simulation should be developed with the following feature:

  —— Correct CSR Force

  —— Highly self-consistent

  —— Low numerical noise

  (Ideally ,Vlasov simulation of phase space distribution, as the Semi-Lagrangian simulation done by R. Walnock, is the direction to go.)