Beam Instability and Microbunching due to Coherent Synchrotron Radiation

Gennady Stupakov & Sam Heifets

SLAC

ICFA workshop on CSR

January 14-18, 2001

Introduction

A relativistic electron beam moving on a circular orbit in free space can radiate coherently if the wavelength of the synchrotron radiation exceeds the length of the bunch. In accelerators coherent synchrotron radiation (CSR) of the bunch is usually suppressed by the shielding effect of the conducting walls of the vacuum chamber. However an initial density fluctuations with a characteristic length much shorter than the bunch length can radiate coherently. If the radiation reaction force results in the growth of the initial fluctuation, one can expect an instability which leads to micro-bunching of the beam and increased coherent radiation at short wavelengths.

Ring

Consider a *coasting* beam moving in a circular orbit of radius R in free space.

$\rho(\delta, z, s)$ — longitudinal distribution function,

$dN = dz \int \rho(\delta, z, s) d\delta$. Vlasov equation

$$\begin{aligned} & \frac{\partial \rho}{\partial s} - \eta \delta \frac{\partial \rho}{\partial z} \\ - \quad & \frac{r_0}{\gamma} \frac{\partial \rho}{\partial \delta} \int_{-\infty}^{\infty} dz' d\delta' W(z - z') \rho(\delta', z', s) = 0 \end{aligned}$$

$\delta = \Delta E / E$

$s = ct$

η – slip factor

r_0 – classical electron radius

$W(z - z')$ – wake function (per unit length of the path).

Neglect the shielding effect, and assume a steady-state wake

$$W(\zeta) = \frac{2}{(3R^2)^{1/3}} \frac{\partial}{\partial \zeta} \frac{1}{\zeta^{1/3}} \quad \text{for } \zeta > 0,$$

and $W(\zeta) = 0$ for $\zeta \leq 0$. The radiation wakefield is localized in front of the moving charge.

The distribution function ρ is a sum of the equilibrium distribution function ρ_0 and a perturbation ρ_1

$$\rho = \rho_0(\delta) + \rho_1(s, \delta, z).$$

with $\rho_1 \ll \rho_0$.

$$n_b = \int \rho_0(\delta) d\delta$$

$$n_1(z, s) = \int \rho_1(\delta, z, s) d\delta$$

Assume

$$\rho_1 = \hat{\rho}_1 e^{-i\omega s/c + ikz}$$

where k is the instability wavenumber

$$(\omega + ck\eta\delta)\hat{\rho}_1 = -i\frac{r_0 c}{\gamma}\frac{\partial \rho_0}{\partial \delta} Z(k) \int d\delta\, \hat{\rho}_1(\delta).$$

Impedance

$$Z(k) = \int_0^\infty d\zeta W(\zeta) e^{-ik\zeta} = iA\frac{k^{1/3}}{R^{2/3}}.$$

The complex factor A is

$$A = 3^{-1/3}\Gamma\left(\frac{2}{3}\right)\left(\sqrt{3}i - 1\right) = 1.63i - 0.94$$

The non-trivial solution of the linearized equation exists if ω satisfies the dispersion relation

$$1 = -\frac{ir_0 c Z(k)}{\gamma}\int \frac{d\delta\,(d\rho_0/d\delta)}{\omega + ck\eta\delta}$$

(cf. with Keil-Schnell)

For a Gaussian distribution function,

$\rho_0 = n_b (2\pi)^{-1/2} \exp(-\delta^2/2\delta_0^2)$

$$\frac{(kR)^{2/3}}{\Lambda} = -\frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dp\, p\, e^{-p^2/2}}{\Omega + p}$$

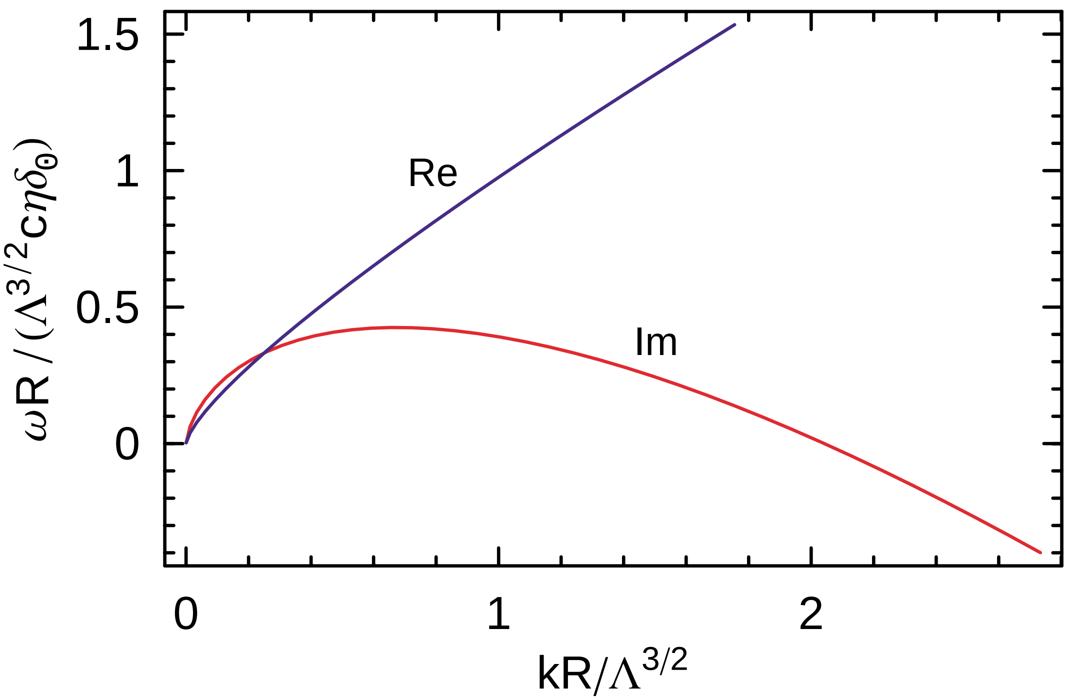
where $\Omega = \omega / ck\eta\delta_0$,

$$\Lambda = \frac{n_b r_0}{\eta \gamma \delta_0^2}$$

Note

$$n_b r_0 = \frac{I}{17 \text{ kA}}$$

Plot of the frequency ω versus k



The beam is unstable for such wavelength that

$$kR < 2.0\Lambda^{3/2}.$$

$$\Lambda \propto \frac{I}{\delta_0^2}$$

The maximum growth rate is reached at $kR = 0.68\Lambda^{3/2}$ and is equal to $\mathrm{Im}\omega_{\mathrm{max}} = 0.43\Lambda^{3/2}c\eta\delta_0/R$.

Assumptions and Limitations

1. Bunched Beam. For a bunched beam of length σ_z the coasting-beam approximation can be applied if $k\sigma_z \gg 1$,

$$\sigma_z \gtrsim 0.5 \frac{R}{\Lambda^{3/2}}$$

2. Shielding. Finite aperture b of the beam pipe — CSR is suppressed due to the shielding effect at

$$kR \lesssim \left(\frac{\pi R}{2b}\right)^{3/2}$$

Hence the instability can only develop for such values of k that $2.0\Lambda^{3/2} > kR \gtrsim (\pi R/2b)^{3/2}$.

$$\frac{R}{b} \lesssim \Lambda.$$

3. Non Circular Orbit. One has to average the wakefield over the beam orbit. This results in an additional factor $R/\langle R\rangle$, where $\langle R\rangle = C/2\pi$

$$\Lambda = \frac{n_b r_0 R}{\eta\gamma\delta_0^2\langle R\rangle}$$

4. Radiation Damping. Effect of the incoherent synchrotron damping in the ring:

$$\mathrm{Im}\omega \to \mathrm{Im}\omega - \gamma_d$$

Numerical Estimates for LER, ALS and VUV rings

Accelerator	LER PEP-II	ALS	VUV NSLS
E (GeV)	3.1	1.5	0.81
η	$1.31 \cdot 10^{-3}$	$1.41 \cdot 10^{-3}$	$2.35 \cdot 10^{-2}$
δ_0	$8.1 \cdot 10^{-4}$	$7.1 \cdot 10^{-4}$	$5.0 \cdot 10^{-4}$
$\langle R \rangle$ (m)	350	31.3	8.11
R (m)	13.7	4	1.91
b (cm)	2	1	2.1
I_b (mA)	2	7.6 (30)	400
σ_z (cm)	1	0.7	4.7
Λ	7	306 $(1.2 \cdot 10^3)$	250
R/b	550	400	90
$R/2\Lambda^{3/2}$ (cm)	1.0	0.037 $(4.7 \cdot 10^{-5})$	0.025

Nonlinear Regime

Assume the perturbation of the beam density $\hat{n}_1 e^{-i\omega s/c+ikz}$. Wake

$$\begin{aligned} w_1 &= \int_{-\infty}^{\infty} dz' W(z-z') n_1(z',s) \\ &= iA\hat{n}_1 \frac{k^{1/3}}{R^{2/3}} e^{-i\omega s/c+ikz}. \end{aligned}$$

Equation for δ

$$\frac{d\delta}{ds} = -\frac{r_0}{\gamma} w_1,$$

$$\delta = \frac{A\hat{n}_1 r_0 c k^{1/3}}{\gamma R^{2/3}\omega} e^{-i\omega s/c+ikz}.$$

When the amplitude of the oscillations becomes comparable to width of the distribution function, $\delta \sim \delta_0$, the linear theory becomes invalid, and one can expect a saturation of the instability.

$$\frac{\hat{n}_1}{n_b} \sim \frac{\gamma \delta_0^2 \eta (kR)^{2/3}}{r_0 n_b} = \frac{(kR)^{2/3}}{\Lambda},$$

(we have used $\omega \sim ck\eta\delta_0$).

Bunch compressor

Similar effects can also occur in a bunch compressor. The effect of microbunching caused by CSR has been observed in computer simulations by Borland (2001), and more recently, in LCLS BC2, by Borland and Emma.

Assumptions

- linear theory
- no shielding
- neglect the transients

Equations

We will use the same coasting beam model as above.

Initial distribution function of the beam

$$\rho_0(\delta, z, 0) = \frac{n_{\rm in}}{\sqrt{2\pi}\delta_0} \exp\left[-\frac{(\delta + uz)^2}{2\delta_0^2}\right]$$

δ_0 – the rms relative energy spread,
$n_{\rm in}$ – number of particles per unit length,
$u = 2\pi V_{\rm rf}/(\lambda_{\rm rf} E)$.

In the compressor $z \to z - R_{56}(s)\delta$ (we assume that $R_{56} > 0$ and $u > 0$).

If there is no wake $d\delta/ds = 0$, $dz/ds = R'_{56}\delta$, and

$$\rho_0(\delta, z, s) = \frac{n_{\rm in}}{\sqrt{2\pi}\delta_0} \exp\left(-\frac{[\delta + u(z - \delta R_{56})]^2}{2\delta_0^2}\right).$$

The beam density at the location s is given by the following equation

$$n_0(s, z) = \int d\delta \rho_0(\delta, z, s) = \frac{n_{\rm in}}{1 - uR_{56}(s)}.$$

With wake

$$\frac{d\delta}{ds} = -\frac{r_e}{\gamma}\int dz' W(z-z')n(s,z'), \quad \frac{dz}{ds} = \delta R'_{56}$$

where $n(s,z) = \int \rho(s,z,\delta)d\delta$.

The Vlasov equation for the beam density $\rho(s,z,\delta)$,

$$\frac{\partial\rho}{\partial s} + \delta R'_{56}\frac{\partial\rho}{\partial z} - \frac{r_e}{\gamma}\frac{\partial\rho}{\partial\delta}\int_{-\infty}^{\infty} dz' W(z-z')n(s,z') = 0$$

For the wake we will assume a steady-state CSR wake.

Linearized equations

Consider a small perturbation ρ_1 of the distribution function

$$\rho(s,z,\delta) = \rho_0(s,z,\delta) + \rho_1(s,z,\delta)$$

The linearized Vlasov equation for ρ_1

$$\begin{aligned} \frac{\partial \rho_1}{\partial s} &+ \delta R'_{56} \frac{\partial \rho_1}{\partial z} \\ &- \frac{r_e}{\gamma} \frac{\partial \rho_0}{\partial \delta} \int dz' d\delta' W(z-z') \rho_1(s,z',\delta') = 0 \end{aligned}$$

Assume a perturbation of this form

$$\rho_1(s,z,\delta) = \hat{\rho}_1 \left(s, \delta(1-uR_{56}(s)) + uz\right) e^{ik(z-\delta R_{56}(s))}$$

where the s dependance should be found from the equations. The perturbation of the bunch density $n_1(s,z) = \int d\delta \rho_1(s,z,\delta)$ is

$$n_1 \propto e^{ikz/(1-uR_{56}(s))}.$$

Wavelength of the modulation $2\pi(1-uR_{56}(s))/k$ decreases during the bunch compression.

Mathematical formulation of the problem

$$n_1(s,z) = \frac{e^{ikz/(1-uR_{56}(s))}}{(1-uR_{56}(s))^{8/3}} G(s)$$

New variable $\xi = R_{56}(s)/(1-uR_{56}(s))$ varies from 0 to ∞ when $R_{56}(s)$ varies from 0 to $1/u$.

$$G(\xi) = G_0(\xi) + (1+u\xi)^{-5/3} \int_0^{\xi} d\xi' M(\xi,\xi') G(\xi')$$

where G_0 is the initial value of G,

$$G_0(\xi) = \frac{1}{(1+u\xi)^{5/3}} \int dp \hat{\rho}_1(0,p) e^{-ikp\xi}$$

and

$$M(\xi,\xi') = A \frac{Q(\xi')}{k^2 R'_{56}(\xi')} (\xi-\xi') e^{-(k\delta_0(\xi-\xi'))^2/2}$$

where

$$Q(s) = \frac{n_{\rm in} r_e}{\gamma k^{2/3} R(s)^{2/3}},$$

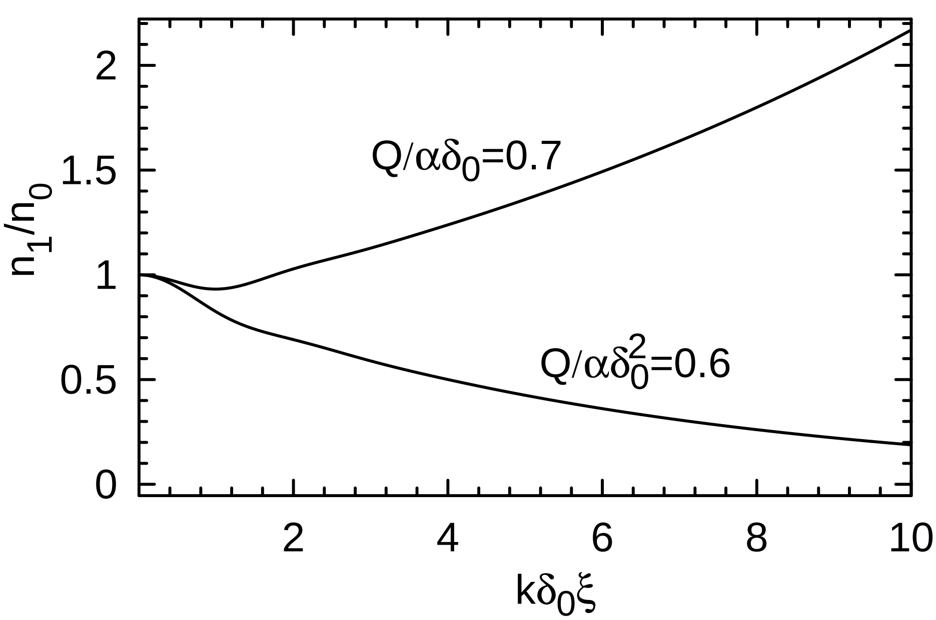
and $A = 1.63i - 0.94$.

In the absence of the wakefield, the is $G = G_0$, and $n_1 \propto \exp(-(k\xi\delta_0)^2/2)$. The wake, however, not only prevents this perturbation from decaying, but can actually amplify it.

Numerical solution

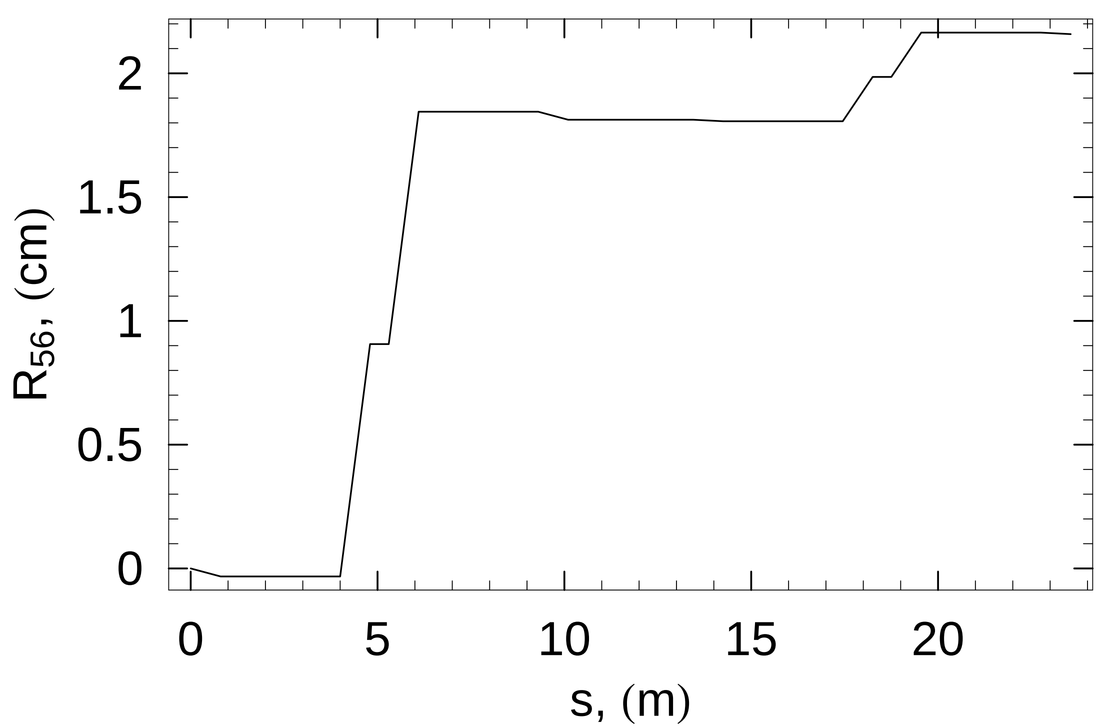
We use a mesh in variable ξ, $0 < \xi < \xi_{\max}$, with $n_{\max} \sim 1000 - 3000$ nodes. G is defined at the mesh points of the grid. The contribution of each subinterval into the integral is evaluated using linear interpolation of G.

The numerical algorithm has been tested for the case $u = 0$ and $R_{56}(s) = \text{const} - \eta s$. This corresponds to the ring case. Instability threshold $Q/\eta\delta_0^2 = 0.62$.



LCLS bunch compressor (old)

The LCLS compressor BC2 consists of eight dipole magnets of length 0.8 m, $E = 4.5$ GeV, the rms bunch length σ_l changes from 195 microns down to 22 microns. Other parameters $\delta_0 = 1.6 \cdot 10^{-5}$, $N = 6.5 \cdot 10^{-9}$, $u = 41\ \mathrm{m}^{-1}$. For the first 4 magnets is $R = 16.2$ m, and the last four $R = 37.2$ m.

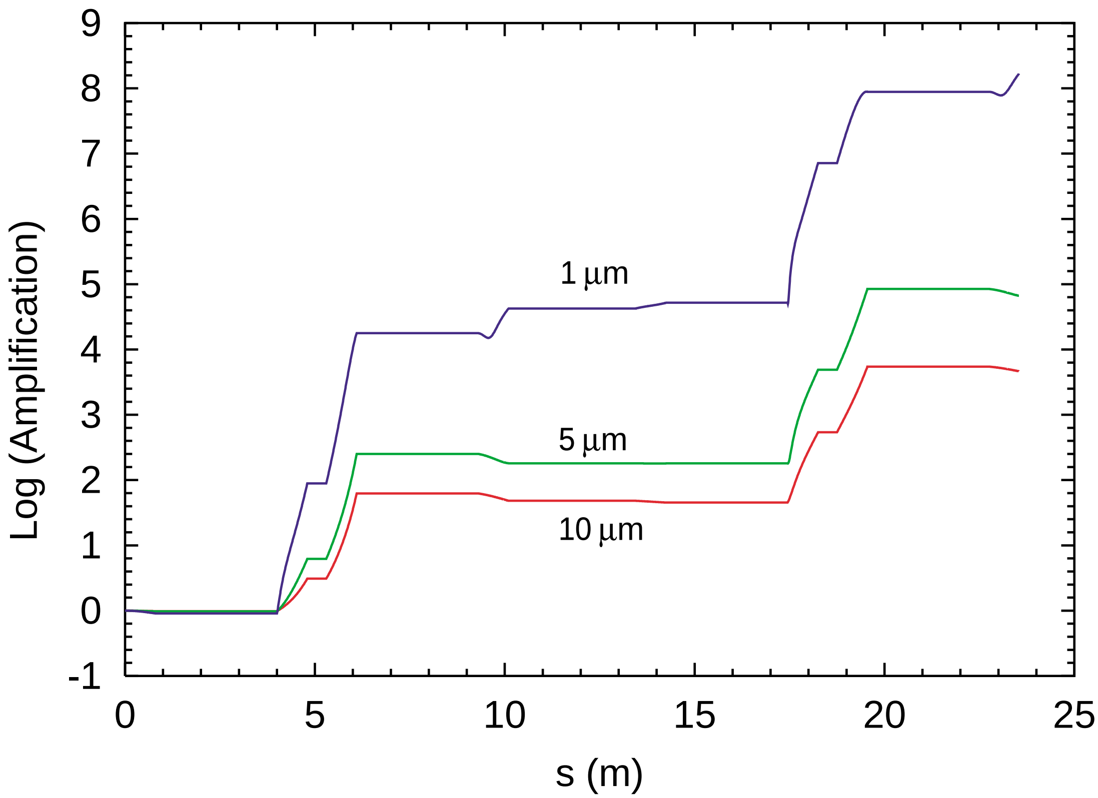


We assumed a Gaussian distribution of the beam density, $n_{\text{in}} = N/\sqrt{2\pi}\sigma_l$. We also assumed initial Gaussian distributions for $\rho_0(\delta)$ and $\hat{\rho_1}(0, \delta)$ with the rms value of δ_0.

At the entrance to the compressor, an initial density perturbation n_1^{init} with the wavelength λ has been specified. The ratio $|n_1(s, z)|/n(s)$ has been calculated throughout the compressor. The amplification factor P

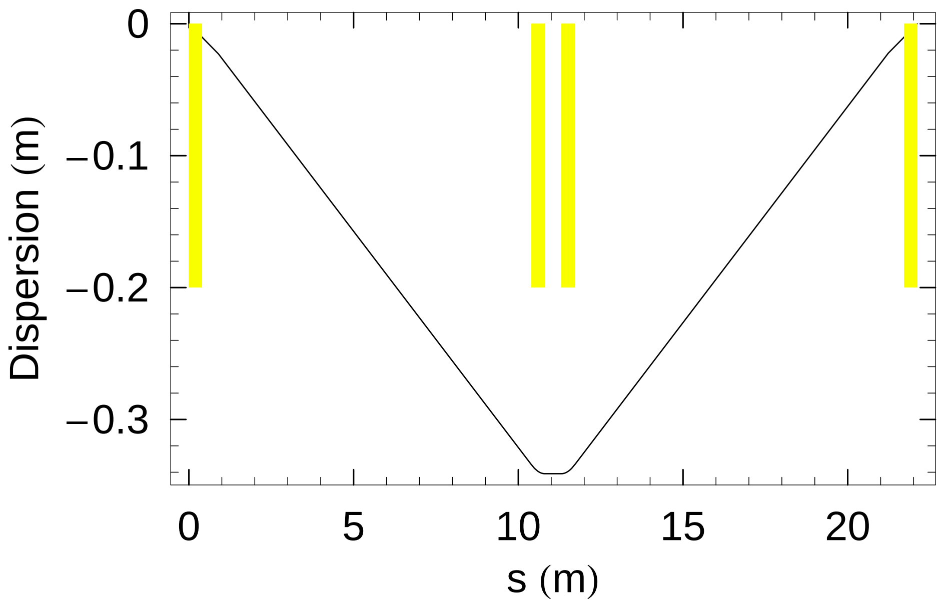
$$P(s) = \frac{|n_1(s, z)|}{n(s)} \frac{n_{\text{in}}}{n_1^{\text{init}}}$$

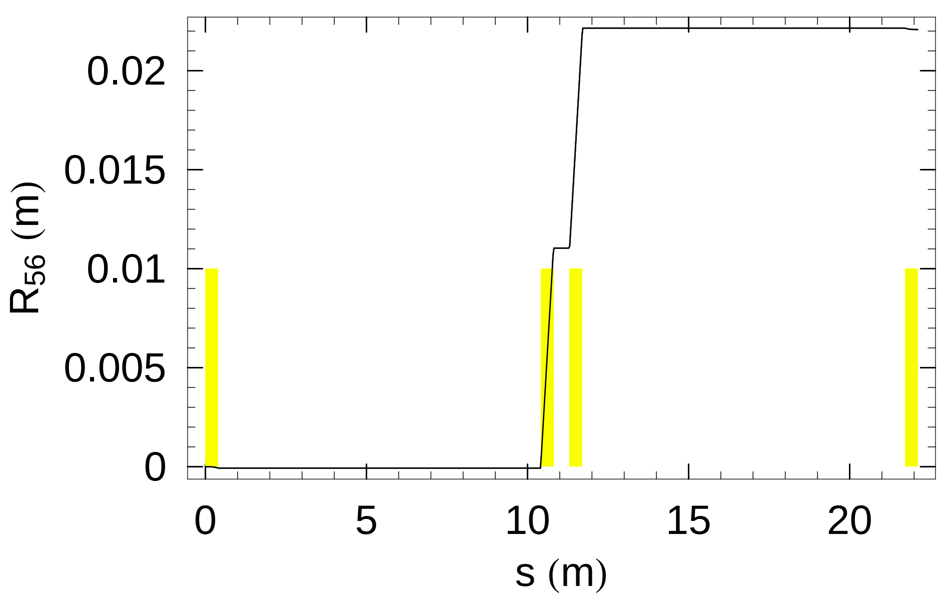
(note that the linear beam density $n(s)$ grows by a factor of $(1 - uR_{56})^{-1} \approx 10$ at the end of the compressor).



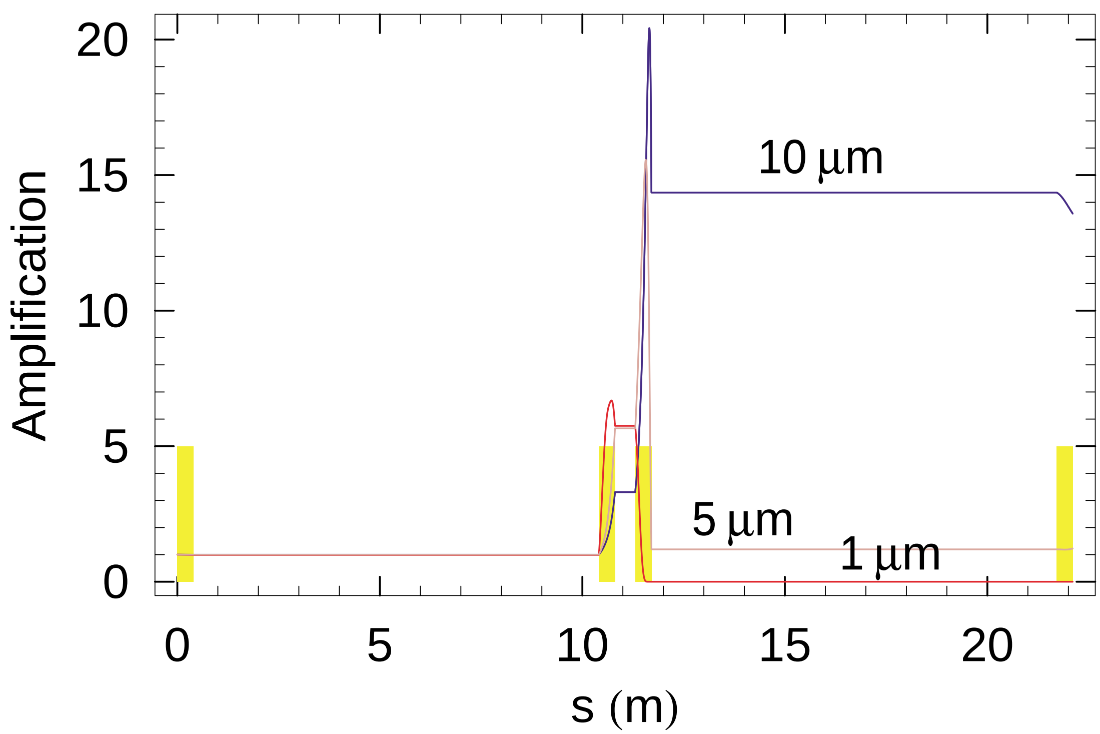
The instability most likely evolves into nonlinear stage.

LCLS bunch compressor (new)





Four dipole magnets of length 0.4 m, $R = 14.2$ m, $E = 4.5$ GeV, the rms bunch length $\sigma_l = 195$ microns, $\delta_0 = 3 \cdot 10^{-5}$, $N = 6.2 \cdot 10^{-9}$, $u = 40$ m^{-1}.



Discussion

To use the notion of the wakefield, the retardation time should be much smaller than the oscillation period.

$$t_{\rm ret} \sim \frac{1}{c} R\psi^{1/3} \sim \frac{R}{c} \frac{1}{(kR)^{1/3}} \ll \omega$$

Putting the frequency from the dispersion relation gives

$$(kR)^{1/3} \ll \frac{\gamma}{r_e n_0 \eta}$$

For typical k

$$\frac{n_0 r_e}{\delta_0 \gamma} \ll 1$$

Conclusion

- CSR induced instability in rings occurs for

$$\Lambda \gtrsim \Lambda_0 \equiv \pi R \left(\frac{\eta\gamma\delta_0^2}{n_b r_e}\right)^{3/2}$$

with a typical growth rate

$$\mathrm{Im}\omega \sim \frac{c\eta\delta_0}{\Lambda_0}.$$

Shielding at these wavelengths would suppress the instability. For a bunched beam, Λ_0 should be $\ll \sigma_z$.

- A linear theory of CSR instability in bunch compressors has been developed. It allows to calculation amplification of an initial sinusoidal modulation of the beam. The results qualitatively agree with computer simulations of LCLS BC2.