2D CSR Wake – Step Beyond A Simple Theory

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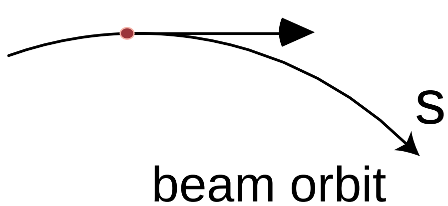
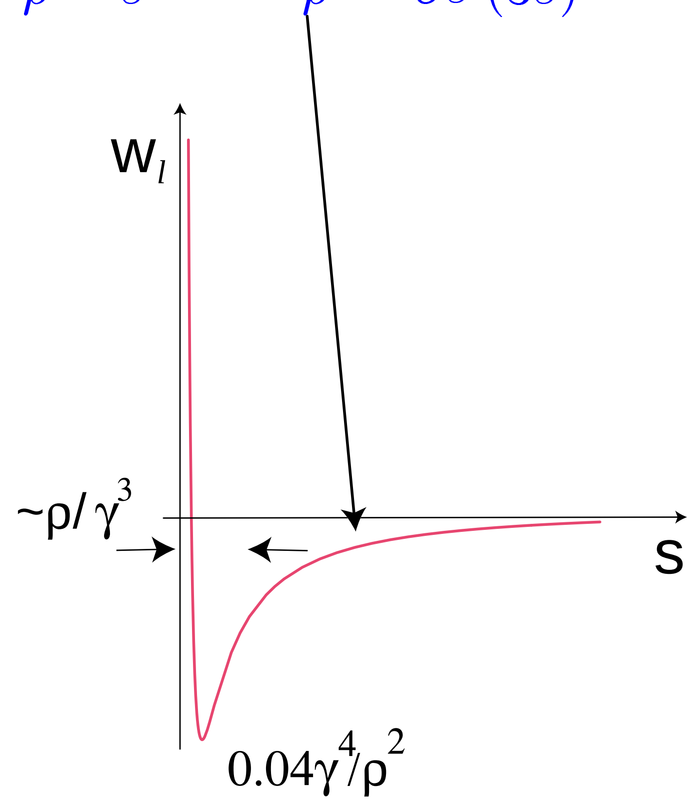
Introduction: a "standard" CSR wake

The standard approach in CSR wake treats the beam as a pencil one, elongated in the direction of motion. All particles move with the same velocity in a circular orbit. In steady state, CSR wake per unit length of path is (Murphy et al., 1995; Derbenev et al. 1995)

$$w_l(s) \approx -\frac{E_{\|}}{q} = -\frac{2}{3^{4/3}\rho^{1/3}s^{4/3}} = \frac{1}{\rho^{2/3}}\frac{\partial}{\partial s}\frac{2}{(3s)^{1/3}}$$





- For $\rho \approx 30$ m and $\gamma \approx 10^4$, $\rho/\gamma^3 = 3 \cdot 10^{-11}$ m
- Shielding effect is negligible if $h \gtrsim s^{2/3}\rho^{1/3}$

Order of magnitude estimate

$$w \sim \frac{1}{\sigma_z^{4/3}\rho^{2/3}}$$

Issues

1. In a bunch compressor, at the final stage, the beam is tilted almost perpendicular to the direction of motion. What is the CSR wake of such beam?

A condition when 2D is small

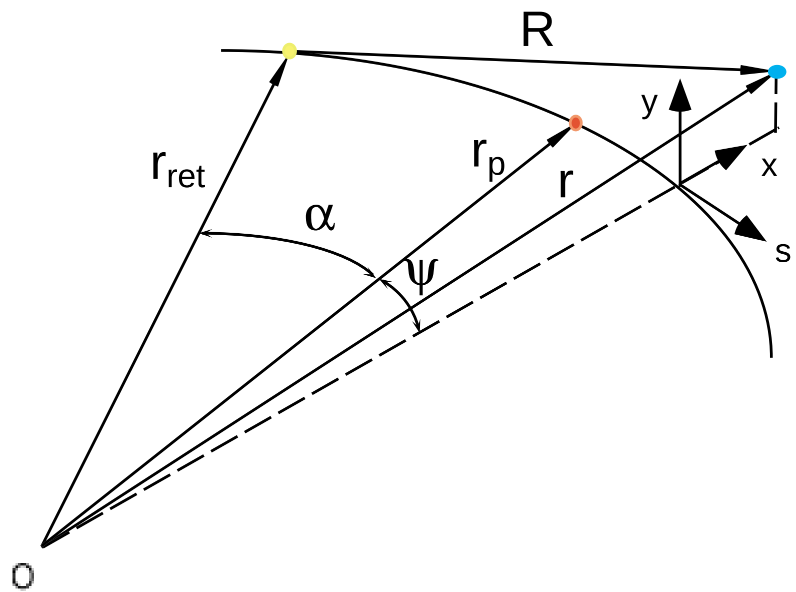
$$\sigma_z \gg \sigma_x \left(\frac{\sigma_x}{\rho}\right)^{1/2}$$

(it may be that $\sigma_z < \sigma_x$)

Physical meaning: transverse coherent scale, $l_\perp \sim \lambda^{2/3}\rho^{1/3} \sim \sigma_z^{2/3}\rho^{1/3}$ should be larger than the transverse beam size σ_x.

Question: what is the wake, when this condition is not satisfied.

2D CSR – Wake: Potentials and Fields



Lienard-Wiechert potentials

$$\varphi\left(\mathbf{r},t\right)=\frac{e}{\left(R-\boldsymbol{\beta}_{\rm ret}\mathbf{R}\right)},\ \mathbf{A}\left(r,t\right)=\frac{e\boldsymbol{\beta}_{\rm ret}}{\left(R-\boldsymbol{\beta}_{\rm ret}\mathbf{R}\right)},$$

$\mathbf{r}$ and t refer to the observation point and time

$\beta_{\rm ret}$ is evaluated at the retarded time $t_{\rm ret}$

$\mathbf{R}=\mathbf{r}-\mathbf{r}_{\rm ret}$, ρ is the bending radius

$\mathbf{r}_{\rm ret}$ is the location of the particle at the retarded time.

The retarded time is determined by the equation $R=c\left(t-t_{\rm ret}\right)$.

Polar coordinate system $(r,\ \theta,\ y)$ has the origin at the center of the orbit. $\mathbf{r} = (\rho + x, \theta_p + \psi, y)$, where $\theta_p = c\beta t/\rho$ is the angle corresponding to the position of the particle at time t.

The equation $R^2 = c^2 \left(t - t_{ret}\right)^2$ determines the angle α

$$\frac{\alpha^2}{\beta^2} = 1 + \left(1 + \frac{x}{\rho}\right)^2 + \frac{y^2}{\rho^2} - 2\left(1 + \frac{x}{\rho}\right)\cos\left(\psi + \alpha\right).$$

In terms of α, ψ and x, the denominator in formulas for potentials is

$$R - \boldsymbol{\beta}_{\text{ret}}\mathbf{R} = \rho\frac{\alpha}{\beta} - \beta\left(\rho + x\right)\sin\left(\psi + \alpha\right),$$

and the polar components of the vector potential are

$$A_r = \frac{e\beta\sin\left(\alpha + \psi\right)}{R - \boldsymbol{\beta}_{\text{ret}}\mathbf{R}}, \quad A_\theta = \frac{e\beta\cos\left(\alpha + \psi\right)}{R - \boldsymbol{\beta}_{\text{ret}}\mathbf{R}}.$$

The electric and magnetic fields are

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\nabla\varphi + \frac{\beta}{\rho}\frac{\partial\mathbf{A}}{\partial\psi}, \ \mathbf{H} = \mathbf{R}\times\frac{\mathbf{E}}{R}.$$

We expand potentials in Taylor series assuming that $\psi \ll 1$ and $x \ll \rho$. Assume that the test particle is in the plane $y = 0$. Assume $\beta = 1$. Use dimensionless x, $x/\rho \to x$.

Expand equation for α and keep lowest order terms

$$-\frac{\alpha^4}{12} + 2\alpha\psi + \alpha^2 x + x^2 = 0$$

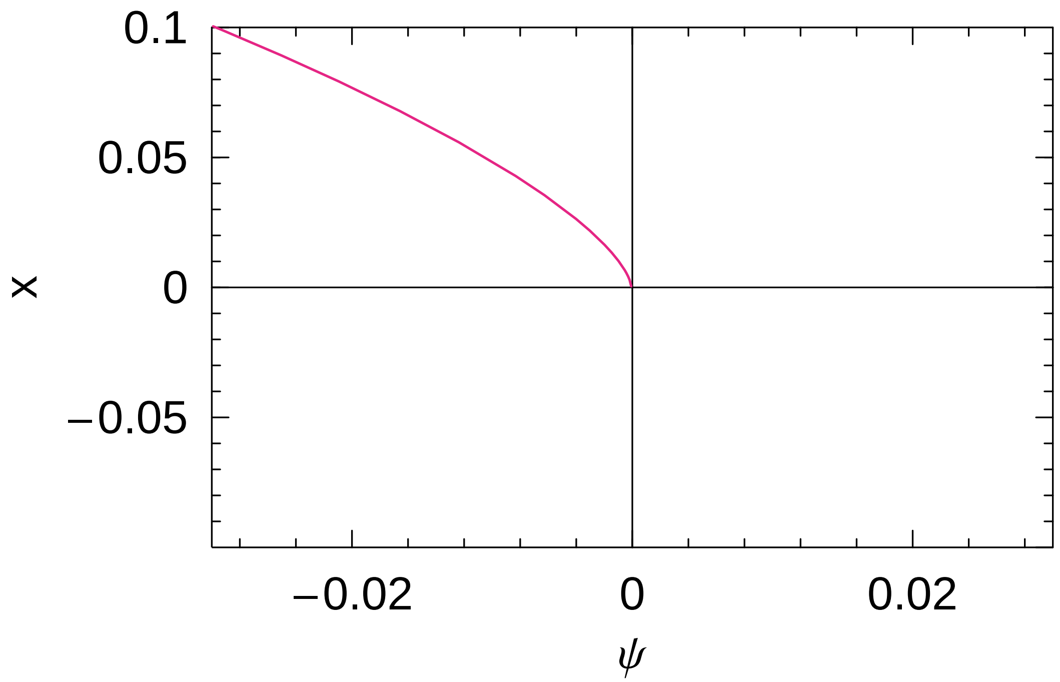
Self-similarity property

$$-\frac{1}{12}\left(\frac{\alpha}{|x|^{1/2}}\right)^4 + 2\frac{\alpha}{|x|^{1/2}}\frac{\psi}{|x|^{3/2}} \pm \left(\frac{\alpha}{|x|^{1/2}}\right)^2 + 1 = 0.$$

$\partial\alpha/\partial\psi \to \infty$ at $\psi = \psi_0$:

$$\psi_0 = -\frac{1}{3}\left(2x\right)^{3/2} = -0.94x^{3/2}, \quad \alpha\left(\psi_0, x\right) = \sqrt{2x}.$$

When x varies from 0 to ∞, the first of the equations determines a curve in the (x, ψ) plane — *a singular line* — where the fields have a singularity. The singularity occurs only because we assume $\beta = 1$.



0.1
0.05
0
−0.05
x
−0.02
0
0.02
ψ

Let us measure φ and A_θ in units e/ρ, and E in units e/ρ^2.

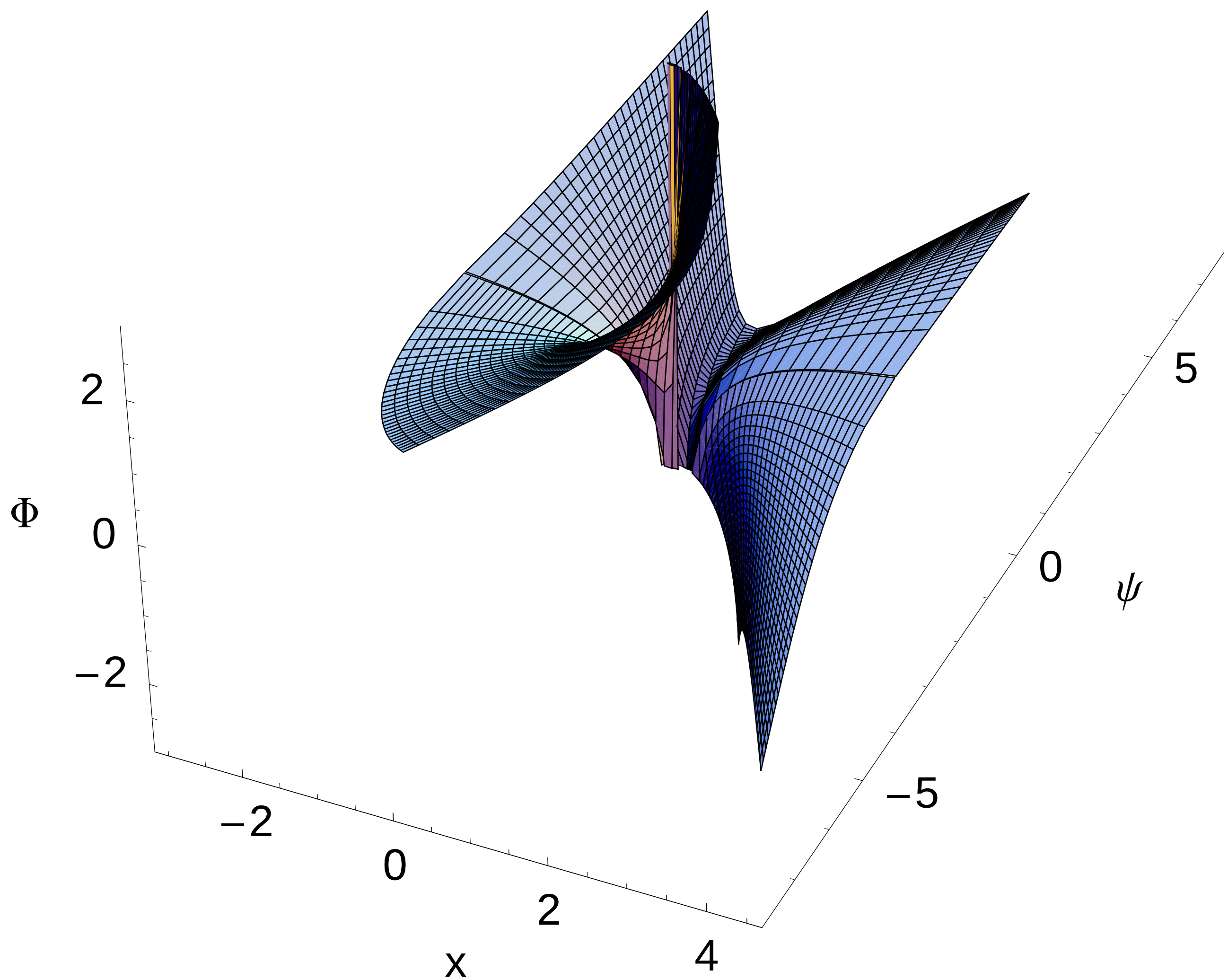
Introduce *longitudinal potential* Φ

$$E_\theta = \frac{\partial \Phi}{\partial \psi}, \;\; \Phi(x,\psi) = -\left(\varphi(1-x) - A_\theta\right)$$

Φ has a scaling property

$$\Phi(x,\psi) = \frac{1}{\sqrt{|x|}}\Pi\left(\frac{\psi}{|x|^{3/2}}\right)$$

where Π has different analytical representations for positive ψ (Π_+) and negative ψ (Π_-).


2
Φ
0
−2
−2
0
2
4
x
5
0
ψ
−5

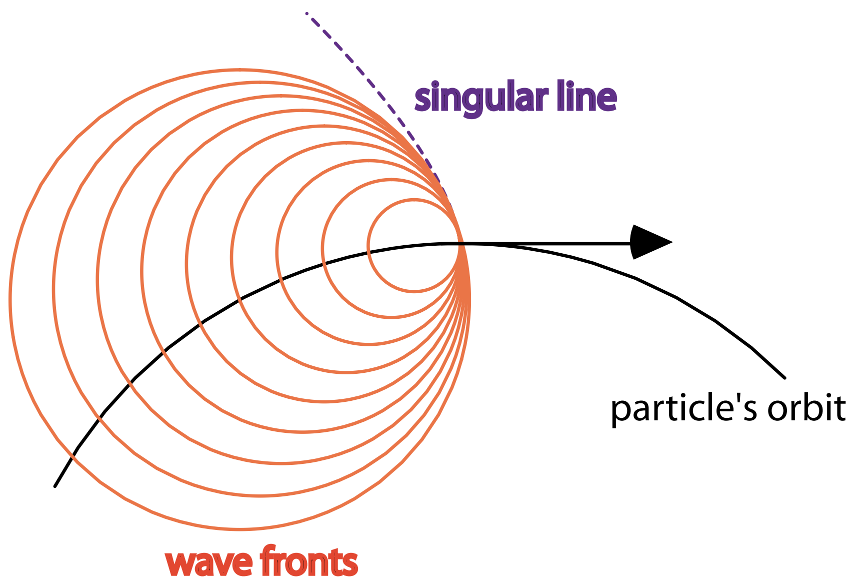
In the vicinity of the singular line

$$\Phi \sim \frac{2^{2/3}}{3^{1/3}(\psi - \psi_0)^{1/3}}$$

$$E_\theta \sim \frac{2^{2/3}}{3^{4/3}(\psi - \psi_0)^{4/3}}$$

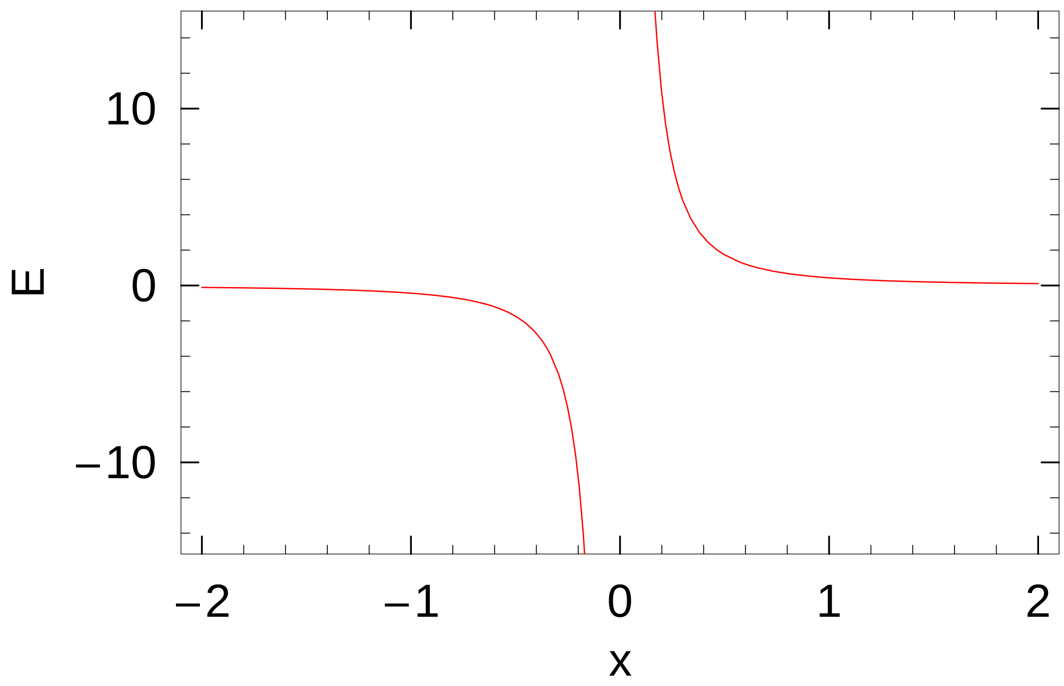
Why singularity at $x \neq 0$?

Draw wavefronts of the radiation emitted at different times by a particle moving with the speed of light around a circle. The wavefronts are condensed on the outer side of the circle and form a caustic. The envelope of the wavefronts in the vicinity of the charge is our singular line.



For $\psi = 0$,

$$E_\theta = \text{sign}(x)\frac{\sqrt{3}}{4x^2}$$



One cannot calculate the wake for a bunch infinitely thin in ψ direction (pencil beam in x-direction).

We now want to calculate the longitudinal force $E_{||}$ acting on a particle of unit charge in a bunch with a two-dimensional distribution function $f\left(x,\psi\right)$,

$$\int f\left(x,\psi\right)dxd\psi = 1.$$

$$E_{||}\left(x,\psi\right) = N\int_{-\infty}^{\infty} dx'd\psi' f\left(x-x',\psi-\psi'\right) E_{\theta}\left(x',\psi'\right)$$

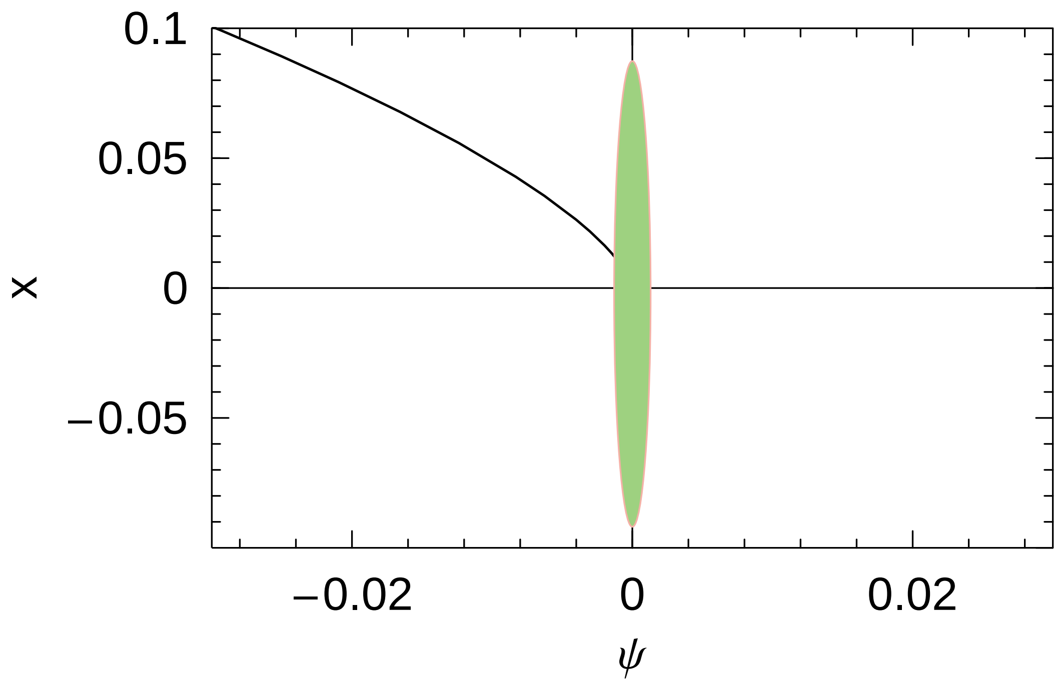
In 2D case, if we use $E_{\theta} = \partial\Phi/\partial\psi$ and integrate by parts, all singularities are integrable:

$$\begin{aligned} E_{||}\left(x,\psi\right) &= N\int_{-\infty}^{\infty} dx'd\psi' f\left(x-x',\psi-\psi'\right)\frac{\partial\Phi\left(x',\psi'\right)}{\partial\psi'} \\ &= -N\int_{-\infty}^{\infty} dx'd\psi'\frac{\partial f\left(x-x',\psi-\psi'\right)}{\partial\psi'}\Phi\left(x',\psi'\right) \end{aligned}$$

Example $(\sigma_\psi = \sigma_z/\rho)$

$$f(x,\psi) = \frac{1}{\sqrt{2\pi}} e^{-\psi^2/2\sigma_\psi^2} \lambda(x)$$

and $\sigma_\psi \ll \Delta x$.



Find the wake integrated over ψ

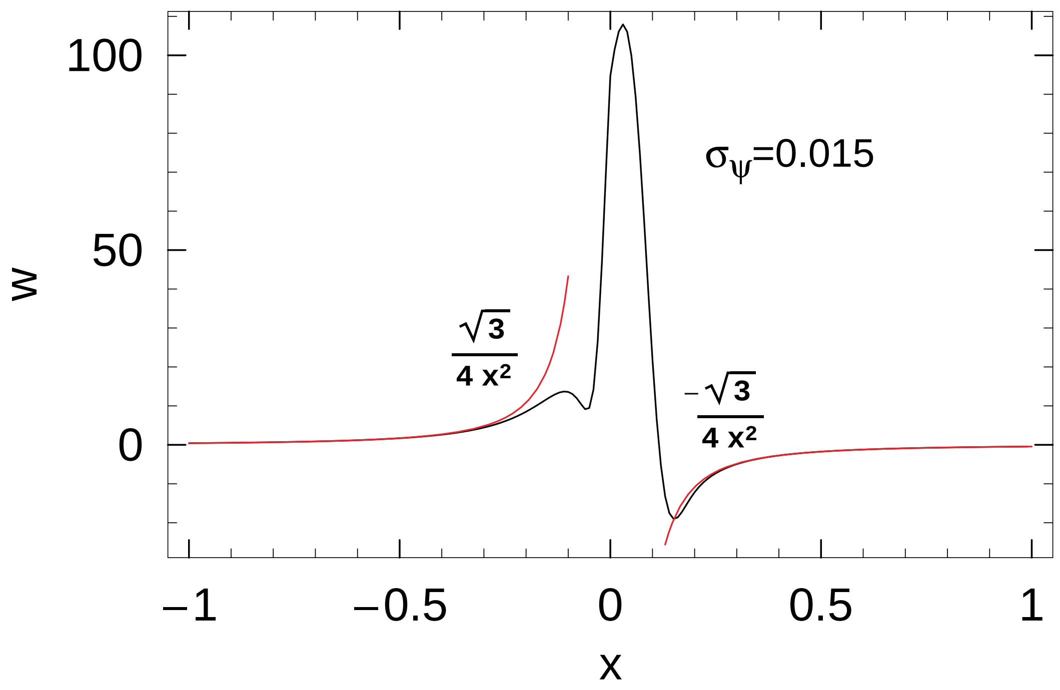
$$\bar{E}_{\|}(x) = \int_{-\infty}^{\infty} d\psi E_{\|}(x,\psi) \frac{1}{\sqrt{2\pi}} e^{-\psi^2/2\sigma_\psi^2}$$

After analytical calculations one finds

$$\bar{E}_{\|}(x) = -\int_{-\infty}^{\infty} dx' w(x-x')\lambda(x')$$

where

$$w(x) = -\frac{x^{5/2}}{4\sqrt{\pi}\sigma_\psi^3} \int_{-\infty}^{\infty} dt\, t[\Pi_+(t) - \Pi_-(t)] e^{-t^2 x^3/4\sigma_\psi^2}$$



The width of the wake is of order of $\sigma_\psi^{2/3}$ and the area under the curve is $S \sim 1/\sigma_\psi^{2/3}$. From numerical integration

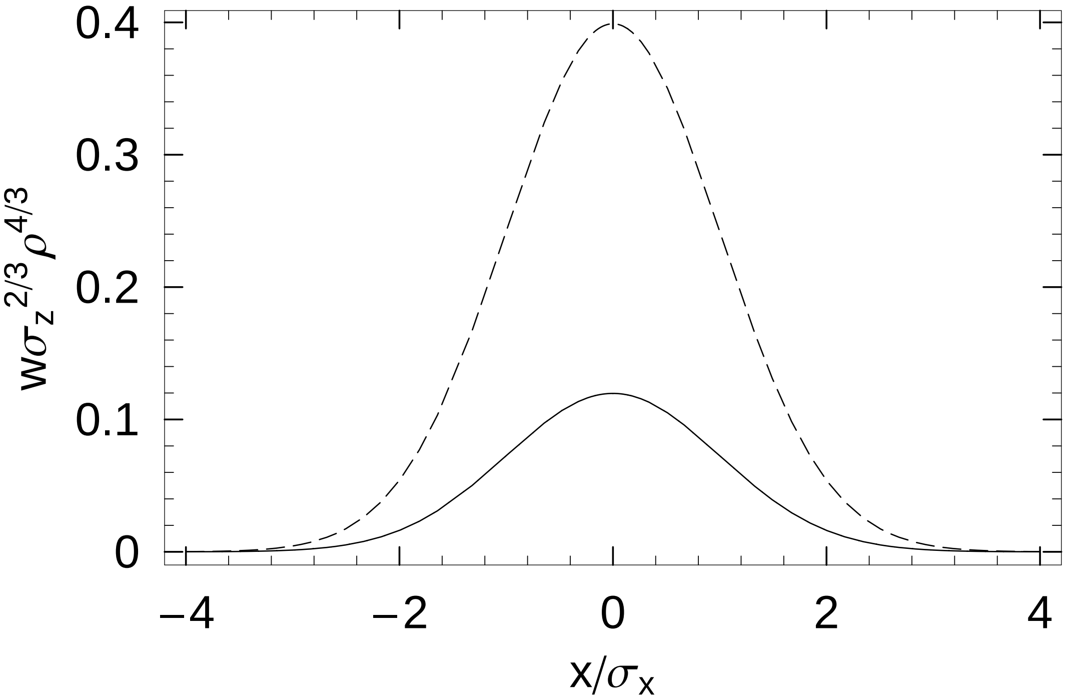
$$S = 0.3/\sigma_\psi^{2/3}$$

If $\Delta x \gg \sigma_\psi^{2/3}$, we can approximate the wake by the δ-function

$$w(x) \approx \frac{0.3}{\sigma_\psi^{2/3}} \delta(x)$$

For a Gaussian distribution in x, $\lambda = (\sqrt{2\pi}\sigma_x)^{-1} \exp(-x^2/2\sigma_x^2)$

$$w_{\mathrm{Gauss}}(x) = \frac{0.3}{\sqrt{2\pi}\sigma_z^{2/3}\sigma_x\rho^{1/3}} \exp(-x^2/2\sigma_x^2)$$



Conclusions

- 2D theory allows to compute the CSR wake for arbitrary 2D distribution of the beam.
- For a Gaussian distribution with σ_z and σ_x old results are valid in the limit

$$\sigma_z \gg \sigma_x \left(\frac{\sigma_x}{\rho}\right)^{1/2}$$

(even when $\sigma_z < \sigma_x$). The wake in this case is of order of

$$w \sim \frac{1}{\sigma_z^{4/3} \rho^{2/3}}$$

In the opposite limit the wake of a beam slice $\sim \delta(x)$ and for a bunch

$$w \sim \frac{1}{\sigma_x^{2/3} \rho^{4/3}}$$

Conclusions (continued)

- At very short distanced, $s < \sigma_x^{3/2}/\rho^{1/2}$, 1D CSR wake saturates due to finite transverse size of the beam.