Introduction to Longitudinal CSR Wake Calculations

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- *The model*
- *Bunch moving on a circle (steady-state CSR)*
- *Transients in a single bend*
- *General small-angle trajectory*

The problem and the model

The problem: find

$$\frac{d\mathcal{E}}{dt}(s, S) = e\vec{v}\vec{E}$$

due to collective field for any particle in a bunch moving on a curved trajectory.

The model:

Given motion

Ultrarelativistic beam:

$$\gamma \gg 1, \quad \beta \simeq 1 - \frac{1}{2\gamma^2}$$

Small-angle approximation: vectors of velocities are within a small cone on a part of trajectory between present position of bunch head and retarded position of the bunch tail

1-D rigid bunch (line charge)

Free space (no shielding)

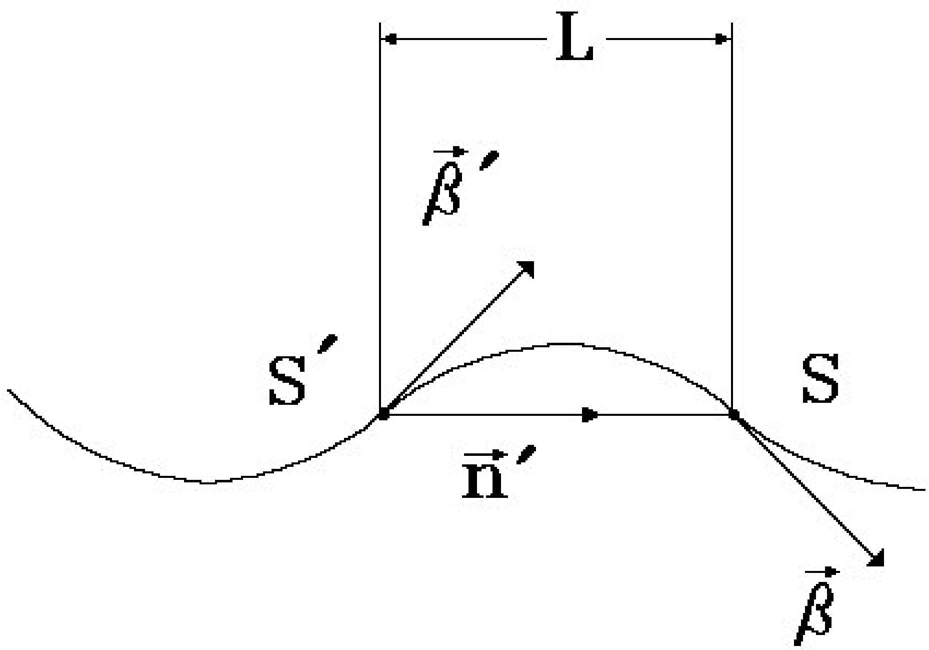
Bunch moving on a circle

Lienard-Wiechert formula

$$\vec{E}(S) = \frac{e}{\gamma^2}\frac{(\vec{n}' - \vec{\beta}')}{L^2\left(1 - \vec{n}'\cdot\vec{\beta}'\right)^3} + \frac{e}{c}\frac{\vec{n}' \times [(\vec{n}' - \vec{\beta}') \times \dot{\vec{\beta}}']}{L\left(1 - \vec{n}'\cdot\vec{\beta}'\right)^3}$$

Test particle is in front of a particle, generating field, at a distance s

$$s = S - S' - \beta L$$



Circle: $S = R\phi$, $S' = R\phi'$, $\psi = \phi - \phi'$

$$s \simeq \frac{R\psi}{2\gamma^2} + \frac{R\psi^3}{24}$$

$$\frac{d\mathcal{E}}{cdt}(s > 0) \simeq \frac{e^2\gamma^4}{R^2}\left\{\frac{1}{(\gamma\psi)^2}\;\frac{3(\gamma\psi)^2 + 4}{[(\gamma\psi)^2/4 + 1]^3} + \frac{1}{2}\;\frac{(\gamma\psi)^2 - 4}{[(\gamma\psi)^2/4 + 1]^3}\right\}$$

Test particle is behind:

$$\frac{d\mathcal{E}}{cdt}(s<0) \simeq -\frac{e^2}{\gamma^2 s^2}$$

Regularization

Straight path:

$$\frac{d\mathcal{E}}{cdt}(s><0) = +-\frac{e^2}{\gamma^2 s^2}$$

Definition of CSR wake:

$$\left(\frac{d\mathcal{E}}{cdt}\right)_{CSR} = \frac{d\mathcal{E}}{cdt} - \left(\frac{d\mathcal{E}}{cdt}\right)_{str}$$

Result:

$$\left(\frac{d\mathcal{E}}{cdt}\right)_{CSR}(s<0) = 0$$

$$\left(\frac{d\mathcal{E}}{cdt}\right)_{CSR}(s>0) = G(s) = \frac{e^2\gamma^4}{R^2}\left\{\frac{1}{2}\,\frac{(\gamma\psi)^2-4}{[(\gamma\psi)^2/4+1]^3}\right.$$

$$\left. +\frac{1}{(\gamma\psi)^2}\left[\frac{3(\gamma\psi)^2+4}{[(\gamma\psi)^2/4+1]^3} - \frac{4}{[(\gamma\psi)^2/12+1]^2}\right]\right\}$$

Small distance $\quad s \ll R/\gamma^3 \quad , \quad \gamma\psi \ll 1$

$$\left(\frac{d\mathcal{E}}{cdt}\right)_{CSR} \simeq -\frac{4e^2\gamma^4}{3R^2} < 0$$

Large distance $\quad s \gg R/\gamma^3 \quad , \quad \gamma\psi \gg 1$

$$\left(\frac{d\mathcal{E}}{cdt}\right)_{CSR} \simeq \frac{32e^2}{R^2\psi^4} \simeq \frac{2e^2}{3^{3/2}R^{2/3}s^{4/3}} > 0$$

CSR wake in a bunch of particles

Bunch with a linear density of particles $\lambda(s)$**:**

$$\frac{d\mathcal{E}}{cdt} = \int\limits_{-\infty}^{s} ds'\lambda(s')G(s-s')$$

Coasting beam $\lambda(s) = \lambda_0$**:**

$$\frac{d\mathcal{E}}{cdt} = \lambda_0 \int\limits_{-\infty}^{s} ds'G(s-s') = \lambda_0 \int\limits_{0}^{\infty} dsG(s) = 0$$

Step function $\lambda(s) = 0$ **for** $s < 0$ **,** $\lambda(s) = \lambda_0$ **for** $s \geq 0$

$$\frac{d\mathcal{E}}{cdt} = \lambda_0 \Phi(s) = \lambda_0 \int\limits_{0}^{s} ds'G(s-s') = \lambda_0 \int\limits_{0}^{s} dxG(x) < 0$$

Arbitrary distribution $\lambda(s)$**:**

$$\frac{d\mathcal{E}}{cdt} = \int\limits_{-\infty}^{s} ds'\frac{d\lambda(s')}{ds'}\Phi(s-s')$$

Radiated power:

$$P = -\left(\frac{d\mathcal{E}}{dt}\right)_b = -\int\limits_{-\infty}^{\infty} ds\lambda(s)\frac{d\mathcal{E}}{dt}(s)$$

Step function

$$\frac{d\mathcal{E}}{cdt} = \lambda_0 \Phi(s) = \lambda_0 \int_0^s ds' G(s') = \lambda_0 \int_0^{\psi_s} d\psi \frac{ds'}{d\psi} G(\psi)$$

$$s = \frac{R\psi_s}{2\gamma^2} + \frac{R\psi_s^3}{24}$$

$$\frac{ds'}{d\psi} = \frac{R}{2\gamma^2} + \frac{R\psi^2}{8}$$

The result:

$$\Phi(\psi_s) = -\frac{4e^2\gamma}{R} \, \frac{(\gamma\psi_s)(8+\gamma^2\psi_s^2)}{(4+\gamma^2\psi_s^2)(12+\gamma^2\psi_s^2)}$$

Rectangular bunch:

$$\lambda_0 = N/l_b \ , \qquad 0 < s < l_b$$

Short bunch $\quad l_b \ll R/\gamma^3$

$$\frac{d\mathcal{E}}{cdt} \simeq -\frac{2\gamma^2 N e^2}{3Rl_b} \ \psi_s = -\frac{4\gamma^4 N e^2}{3R^2 l_b} \ s$$

Radiated power (Larmor formula)

$$P = -\frac{N}{l_b} \int_0^{l_b} ds \frac{d\mathcal{E}}{dt}(s) = \frac{2\gamma^4 N^2 e^2 c}{3R^2}$$

Long bunch $\quad l_b \gg R/\gamma^3$

$$\frac{d\mathcal{E}}{cdt} = \frac{N}{l_b}\Phi(s) \simeq -\frac{N}{l_b}\,\frac{4e^2}{R}\,\frac{1}{\psi_s} = -\frac{N}{l_b}\,\frac{2e^2}{3^{1/3}R^{2/3}s^{1/3}}$$

Radiated power:

$$P \simeq \frac{3^{2/3}N^2e^2c}{R^{2/3}l_b^{4/3}}$$

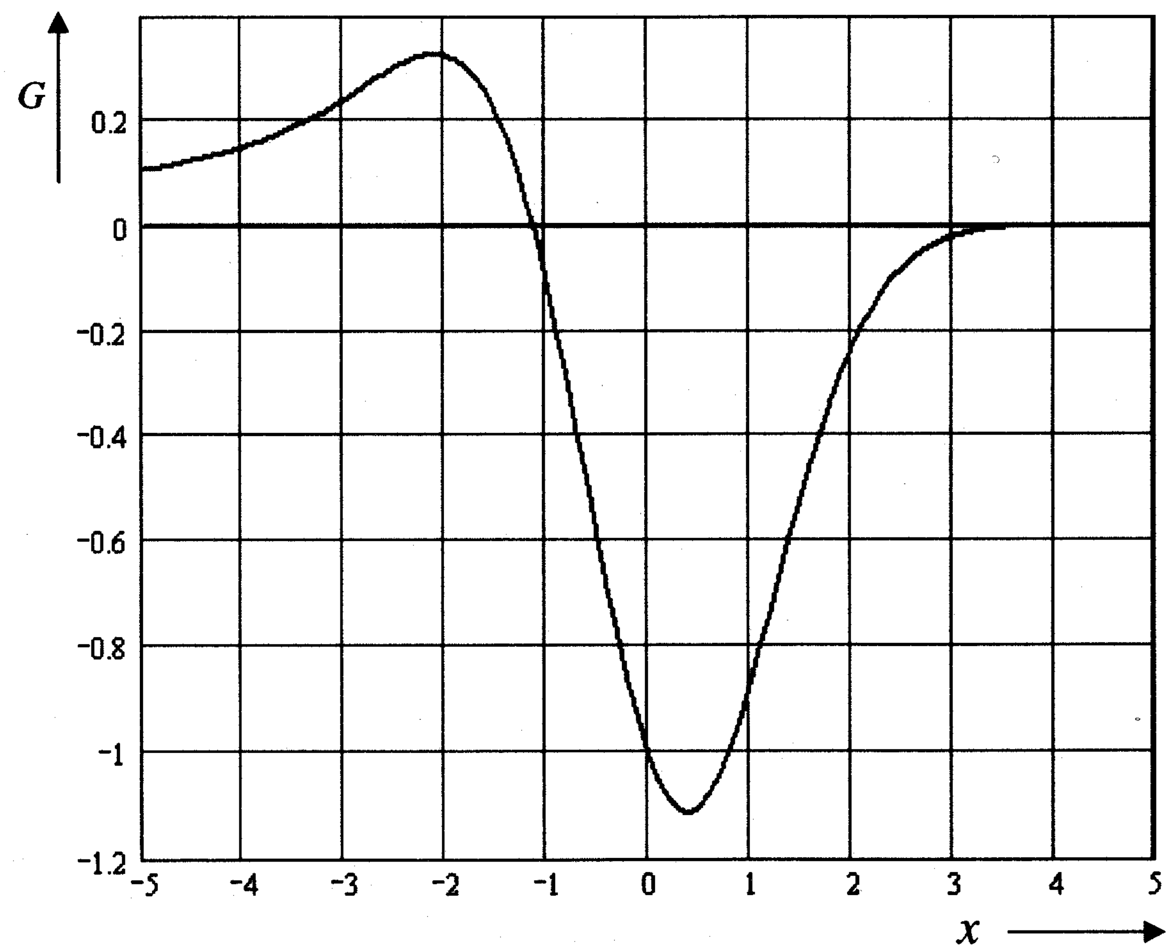
Arbitrary profile , long bunch:

$$\frac{R}{\gamma^3}\frac{d\lambda(s)}{ds} \ll \lambda(s)$$

$$\frac{d\mathcal{E}}{cdt} \simeq -\frac{2e^2}{3^{1/3}R^{2/3}}\int_{-\infty}^{s}\frac{ds'}{(s-s')^{1/3}}\frac{d\lambda(s')}{ds'}$$

$$E_c = \frac{1}{\sqrt[3]{3}(2\pi)^{3/2}} \frac{1}{R_0^{2/3}\sigma^{4/3}} \frac{q}{\varepsilon_0}$$

$$G(w) = \sqrt{2\pi} \int_0^{\infty} \frac{g'(x+w)}{\sqrt[3]{x}} dx$$



Gaussian profile:

$$\lambda(s) = \frac{N}{(2\pi)^{1/2}\sigma}\ exp\left[-\frac{s^2}{2\sigma^2}\right]$$

$$\left(\frac{d\mathcal{E}}{cdt}\right)_{CSR} \simeq -\frac{2Ne^2}{(2\pi)^{1/2}3^{1/3}R^{2/3}\sigma^{4/3}}\ F\left(\frac{s}{\sigma}\right)$$

$$F(x) = \int_{-\infty}^{x} \frac{dx'}{(x-x')^{1/3}}\frac{d}{dx'}e^{-(x')^2/2}$$

$$P \simeq \frac{3^{1/6}N^2e^2c}{2\pi R^{2/3}\sigma^{4/3}}\ \left[\Gamma\left(\frac{2}{3}\right)\right]^2$$

Comparison with far zone

Power spectral density of coherent radiation:

$$\frac{dP_{coh}}{d\omega} = N^2\eta(\omega)\frac{dP_1}{d\omega}$$

Rectangular bunch:

$$\eta(\omega) = \left(\sin\frac{\omega l_b}{2c}\right)^2\left(\frac{\omega l_b}{2c}\right)^{-2}$$

Long bunch:

$$l_b \gg R/\gamma^3 \ , \qquad \omega \ll \omega_c \simeq c\gamma^3/R$$

For a single electron:

$$\frac{dP_1}{d\omega} \simeq \frac{9\Gamma(2/3)3^{5/6}}{4\pi 2^{2/3}}\ \frac{e^2(\omega/c)^{1/3}}{R^{2/3}}$$

CSR power:

$$P_{coh} = \frac{3^{2/3}N^2e^2c}{R^{2/3}l_b^{4/3}}$$

An order of magnitude estimates:

$$\frac{d\gamma}{cdt} \simeq \frac{I}{I_A}\frac{1}{L_F} \ , \qquad L_F \simeq (24\sigma R^2)^{1/3} \ , \qquad I_A = 17 \ kA$$

$$R = 10 \ \mathbf{m} \quad , \quad L_d = 0.5 \ \mathbf{m}$$

$$\sigma = 200 \ \mu\mathbf{m} \quad , \quad I = .6 \ kA \qquad L_F \simeq 0.8 \ \mathbf{m} \quad , \quad \delta\gamma \simeq 0.02 \quad (10 \ keV)$$

$$\sigma = 20 \ \mu\mathbf{m} \quad , \quad I = 6 \ kA \qquad L_F \simeq 0.4 \ \mathbf{m} \quad , \quad \delta\gamma \simeq 0.4 \quad (200 \ keV)$$

Transients in a single bend

CSR wake exists when a test particle is inside or behind the magnet and retarded positions are inside or in front of the magnet

The following property holds for a coasting beam:

$$\frac{d\mathcal{E}}{cdt} = \lambda_0 \int_{-\infty}^{s} ds' G(s - s') = \lambda_0 \int_{0}^{\infty} ds G(s) = 0$$

One can simplify calculations for the step function

$$\Phi(s) = \int_{0}^{s} ds' G(s - s') = -\int_{-\infty}^{0} ds' G(s - s') = -\int_{s}^{\infty} dx G(x)$$

Complementary trajectory may not coincide with the actual one. One can take, for instance, a trajectory tangential to the actual one at the retarded position of the tail

The results are confirmed by comparison with far zone

Scaling:

Bunch length to be compared with R/γ^3

Bending angle ϕ_m **with** $1/\gamma$

Magnet length $R\phi_m$ **to be compared with the steady-state formation length** $R\psi_s$

$$s = \frac{R\psi_s}{2\gamma^2} + \frac{R\psi_s^3}{24}$$

In most practically interesting cases bunch is long, magnet is long. In such cases we should only compare ϕ_m **and** $\psi_s \simeq (24s/R)^{1/3}$

Transients in the case $s \gg R/\gamma^3 \quad , \quad \phi_m \gg \psi_s \simeq (24s/R)^{1/3}$

Entrance

$$\Phi(s,\phi) \simeq 0 \qquad for \qquad \phi < \frac{\psi_s}{2^{2/3}}$$

$$\Phi(s,\phi) \simeq -\frac{4e^2}{R\phi} \qquad for \qquad \frac{\psi_s}{2^{2/3}} < \phi < \psi_s$$

$$\Phi(s,\phi) \simeq -\frac{4e^2}{R\psi_s} \qquad for \qquad \phi > \psi_s$$

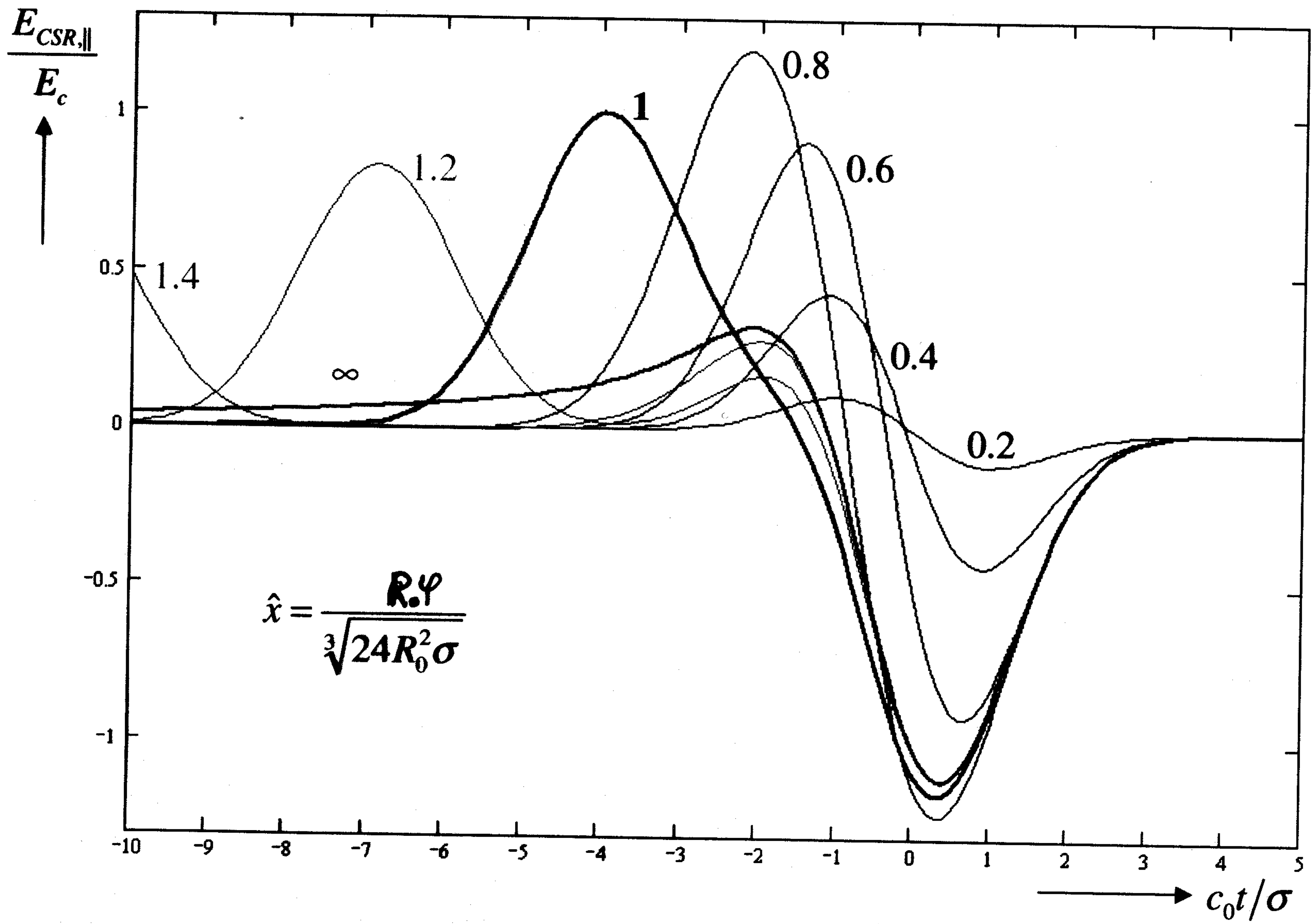
At $\phi = \psi_s/2^{2/3}$ **the retarded position of tail is at the distance** $(3/4)^{2/3}\gamma(s^2R)^{1/3}$ **from the magnet**

Arbitrary profile:

$$\frac{d\mathcal{E}}{cdt}(s,\phi) = \int_{-\infty}^{s} ds' \frac{d\lambda(s')}{ds'}\Phi(s-s',\phi) = -\int_{s-R\phi^3/6}^{s-R\phi^3/24} ds' \frac{d\lambda(s')}{ds'}\left(\frac{4e^2}{R\phi}\right)$$
$$-\frac{2e^2}{3^{1/3}R^{2/3}} \int_{s-R\phi^3/24}^{s} \frac{ds'}{(s-s')^{1/3}}\frac{d\lambda(s')}{ds'}$$

$$= -\frac{2e^2}{3^{1/3}R^{2/3}} \left\{ \left(\frac{24}{R\phi^3}\right)^{1/3} \left[\lambda\left(s-\frac{R\phi^3}{24}\right) - \lambda\left(s-\frac{R\phi^3}{6}\right)\right] + \int_{s-R\phi^3/24}^{s} \frac{ds'}{(s-s')^{1/3}}\frac{d\lambda(s')}{ds'} \right\}$$

Transient CSR Field: Injection



Estimate of formation length $L_F \simeq \gamma(\sigma^2 R)^{1/3}$

$$R = 10\ \mathbf{m} \quad , \quad L_d = 0.5\ \mathbf{m} \quad , \quad \Delta L = 5\ \mathbf{m} \quad , \quad \sigma = 200\ \mu\mathbf{m}$$

$L_F \simeq \gamma \times 7\ \mathbf{mm}$

$\mathcal{E} = 500\ \mathbf{MeV}$ $\qquad L_F \simeq 7\ \mathbf{m}$

$\mathcal{E} = 5\ \mathbf{GeV}$ $\qquad L_F \simeq 70\ \mathbf{m}$

Exit

$$\Phi(s,x) \simeq -\frac{4e^2}{R\Delta\phi(s,x)+2x} + \frac{e^2}{s\gamma^2} \qquad for \qquad x < 2\gamma^2 s$$

$\Delta\phi$ **is the solution of quartic equation**

Behaviour at different x

$x \simeq R\psi_s \quad , \quad \Delta\phi \simeq \psi_s$

$$\Phi(s,x) \simeq -\frac{4e^2}{R\Delta\phi(s,x)+2x}$$

$R\psi_s \ll x \ll s\gamma^2 \quad , \quad \Delta\phi \simeq \psi_s$

$$\Phi(s,x) \simeq -\frac{2e^2}{x}$$

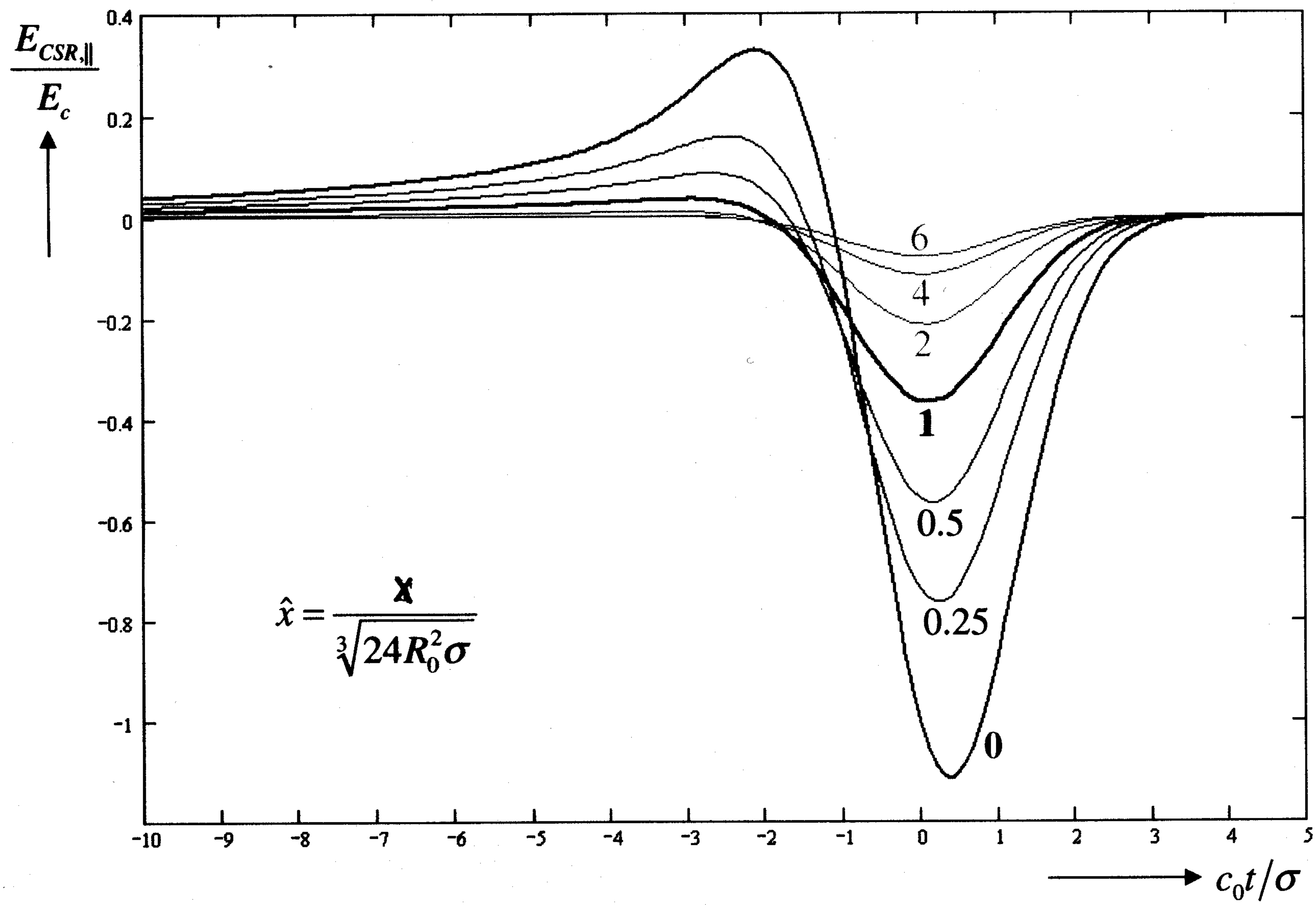
$x \simeq s\gamma^2 \quad , \quad \Delta\phi \simeq \psi_s$

$$\Phi(s,x) \simeq -\frac{2e^2}{x} + \frac{e^2}{s\gamma^2}$$

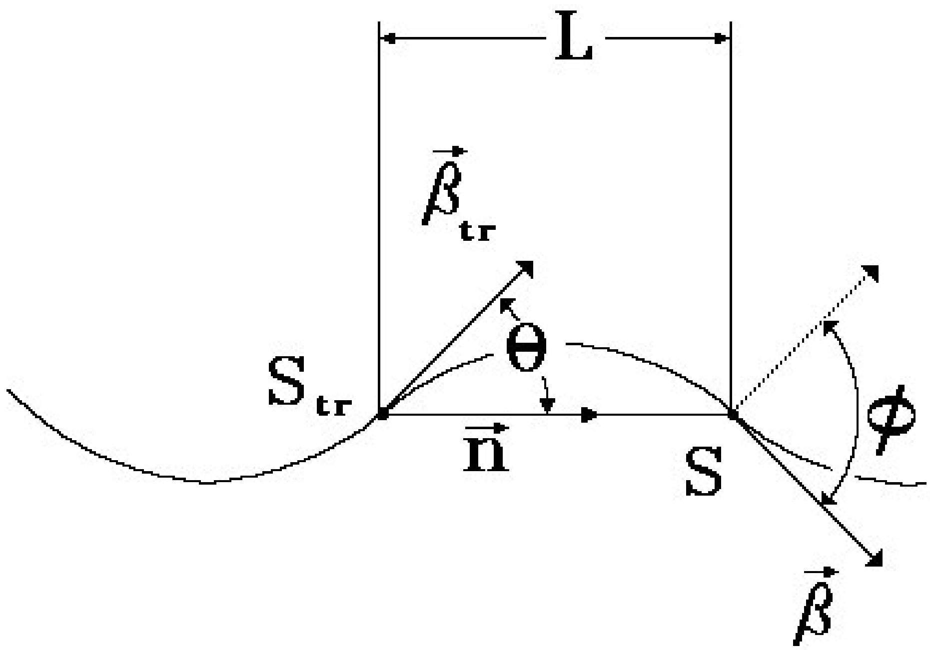
Wake is weak but distance is large

$$\delta\gamma \simeq \frac{I}{I_A}\; ln\left(\frac{l_b\gamma^3}{R}\right)$$

Transient CSR Field: Ejection



A plane small-angle trajectory

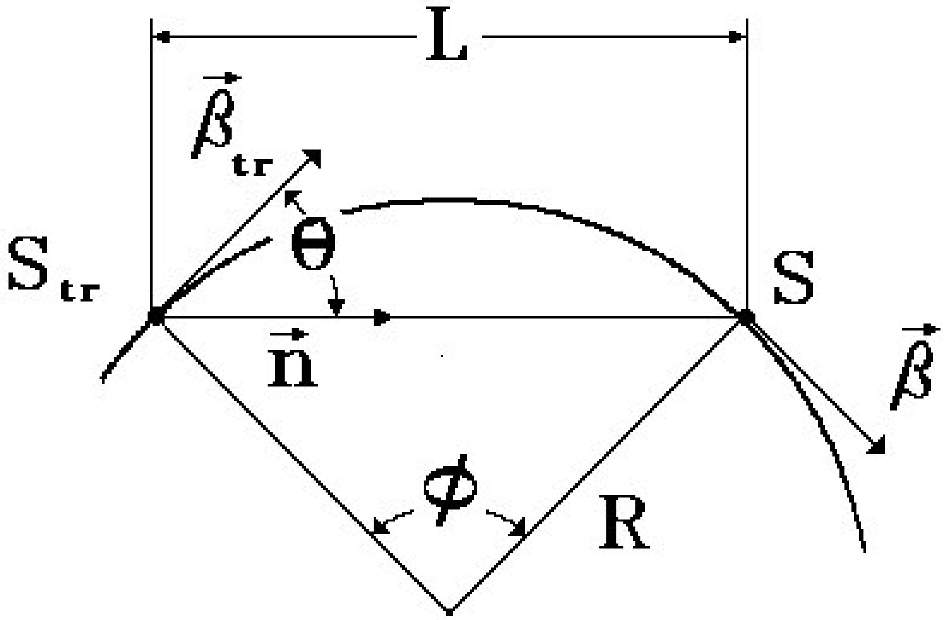


$$s = S - S_{tr} - \beta L$$

$$\Phi(s, S) = e^2 \left(\frac{1}{\gamma^2 s} - \frac{2}{L} \, \frac{1 + \gamma^2 \phi\theta}{1 + \gamma^2 \theta^2} \right)$$

$$\frac{d\mathcal{E}}{cdt}(s, S) = \int\limits_{-\infty}^{s} ds' \frac{d\lambda(s')}{ds'} \Phi(s - s', S)$$

An example: circular motion



$$S - S_{tr} = R\phi$$

$$L = 2R\sin(\phi/2) \simeq R\phi - R\phi^3/24$$

$$s \simeq \frac{R\phi}{2\gamma^2} + \frac{R\phi^3}{24}$$

$$\theta = \phi/2$$

$$\Phi(s) = e^2\left(\frac{1}{\gamma^2 s} - \frac{2}{R\phi}\,\frac{1+\gamma^2\phi^2/2}{1+\gamma^2\phi^2/4}\right) = -\frac{4e^2\gamma}{R}\frac{(\gamma\phi)(8+\gamma^2\phi^2)}{(4+\gamma^2\phi^2)(12+\gamma^2\phi^2)}$$

$$\Phi(s) \simeq -\frac{4e^2}{R\phi} = -\frac{2e^2}{3^{1/3}R^{2/3}s^{1/3}} \qquad for \quad s \gg R/\gamma^3$$

$$\frac{d\mathcal{E}}{cdt}(s) = -\frac{2e^2}{3^{1/3}R^{2/3}}\int_{-\infty}^{s}\frac{ds'}{(s-s')^{1/3}}\frac{d\lambda(s')}{ds'}$$