CSR-INDUCED ENERGY SPREAD IN THE FRAMEWORK OF A SMALL ANGLE APPROXIMATION

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Abstract

In the present contribution we address the energy spread induced by Coherent Synchrotron Radiation in a bunch of relativistic electrons following a curved trajectory in vacuum without shielding. Our considerations include an innovative feature; namely, we assume a privileged direction of motion for the electrons by considering the transverse velocity small. This results in a consistent use of a small-angle approximation, which eventually makes the computation of the collective effects more flexible and intrinsically more efficient, to be possibly employed in a self-consistent simulation framework. Computational results will be reported and compared with results obtained by other authors. Further on, with the help of our approach, we study the transient between a straight and a circular path in the low-γ region, where the CSR longitudinal force is energy-dependent, extending to this case one of the results obtained in [1].

1 INTRODUCTION

Very short, high-charge bunches of electrons will be used in the next generation light sources. Bunch compression chicanes are expected to be often used in order to provide very high peak-current beams for X-ray SASE-FELs. However, their production and utilization may prove difficult due to Coherent Synchrotron Radiation. CSR has been a matter of active theoretical [1], [3], [4], [5], [6], [7], numerical [8], [9],[10], [11] and experimental [12], [13] research in the past few years. Measurements and computations are in reasonable agreement.

In the present paper we address the problem of energy spread induced by radiative collective interactions within a short electron bunch following its trajectory in vacuum without shielding. The novel feature of our consideration is that we consistently apply a small-angle approximation, a natural technique for ultrarelativistic particles. By this, the efforts necessary for the treatment of an arbitrary trajectory is considerably reduced; on the other hand, the class of allowed trajectories is somewhat restricted.

Eventually, this route is expected to lead to an efficient computational tool for the design of magnetic systems for high-peak-current electron bunches, to be possibly used in the framework of a self-consistent simulation. A more exhaustive description of our method can be found in [14].

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Computational results are reported, and compared to the ones obtained by other authors. Moreover, with the help of our approach, we consider the problem of a transient between a straight path and a region of constant bending radius. We investigate, in particular, the low-energy case, where the CSR longitudinal force is γ-dependent. Such an investigation finally leads to an extension of the formula in [1] for the rate of energy change of an electron as a function of its position in a Gaussian bunch at a fixed time (that is, at a fixed position within the magnet).

Besides this introduction the work is organized as follows. In Section 2 we describe the small angle approximation, we use it to deal with a system of two electrons, and we evaluate the energy change for a test particle in a bunch. In Section 3 computational results are reported, while low-γ transient simulations are presented in Section 4; there, a generalization of the formula for the rate of energy change of an electron reported in [1] is given too. Finally, in Section 5, we come to conclusions and speculations.

2 THE SMALL-ANGLE APPROXIMATION METHOD

We will consider the bunch as a 'rigid', 1D, charged object with a given linear charge density distribution, together with [1], [4], [15], [5]. We define a cartesian reference frame (x, y, z) as shown in Fig. 1, where the z-axis coincides with the direction of the initial velocity.

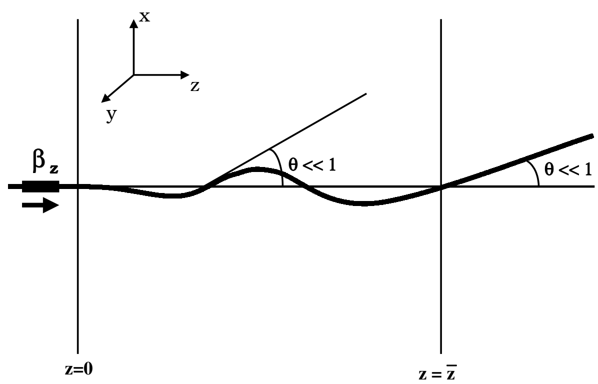

Figure 1: Schematic of a particle trajectory in the small angle approximation

We will assume that, before and after the magnets ($z < 0$ or $z > \overline{z}$), the bunch moves along a rectilinear path with constant velocity, while inside the magnetic system ($0 < z < \overline{z}$) it follows a path subject, in the spirit of the small-angle approximation, to the only constraint that the angle θ formed by the velocity vector with the z-axis is always small, i.e. $\theta \ll 1$. Note that θ can still be small or large as compared to the other small parameter of the problem, γ^{-1} (that is $\gamma \gg 1$), where γ is the usual Lorentz factor.

It is quite natural to neglect differences in transverse velocities of the electrons, that is $l_b(dv_{x,y}/dz) \ll v_{x,y}$, where l_b is the longitudinal extent of the bunch and $v_{x,y}(z)$ are the components of the transverse velocity.

Moreover we will consider the bunch moving on a fixed trajectory by assuming zero initial energy spread and no change of particles energy during the passage of the bunch through the magnetic system. The back influence of radiative effects on the motion of particles is therefore neglected: the consistency of this assumption must, of course, be verified *a posteriori*.

Let us define the local particle velocity $\mathbf{v}$ and a unit vector $\hat{\mathbf{n}}$ connecting two points lying on the same trajectory. One has to distinguish explicitly between their longitudinal and transverse components, assuming the latter to be small, according to the small-angle approximation. Keeping first and second order terms one gets the following well known expressions:

$$n_z \simeq 1 - \frac{1}{2}\vec{n_\perp}^2, v_z \simeq c\left(1 - \frac{1}{2\gamma^2}\right) - \frac{\vec{v_\perp}^2}{2c} . \quad (1)$$

Once the bunch trajectory is known, the problem of radiative collective effects within the bunch reduces to properly accounting for signal retardation in pairwise interactions between individual electrons. Let us consider a test particle inside the bunch. Its present velocity and its present position in the laboratory frame of reference will be denoted as $\vec{v_0}(t)$ and $\vec{r_0}(t)$, respectively. We are interested in its interaction with some other bunch particle −the source particle− whose present position will be denoted as $\vec{r}(t)$. Causality defines the well-known retardation condition between the two particles:

$$|\vec{r_0}(t) - \vec{r}(t')| = c\,(t - t') \ , \quad (2)$$

where $\vec{r}(t')$ denotes the retarded position of the source particle (t' being the so called retarded time), and $(t - t')$ is the time delay associated with signal propagation.

The small-angle approximation considerably simplifies the treatment of the above retardation condition.

In fact, under our assumptions, t and z are uniquely mapped onto each other that makes it possible to switch from a retardation condition expressed in function of time to one in z:

$$\frac{(z_0 - z')}{\gamma^2} + \int_{z'}^{z_0} d\zeta \vec{\beta_\perp}^2(\zeta) - \frac{1}{(z_0 - z')}\left(\int_{z'}^{z_0} d\zeta \vec{\beta_\perp}(\zeta)\right)^2 \simeq 2\Delta z \ , \quad (3)$$

where Δz, to be regarded as a constant, is the z-projected distance between the present positions of the source and of the test particle

$$\Delta z = z_0 - z. \quad (4)$$

Note that for $\Delta z > 0$ the position of the source particle is *behind* that of the test particle: as has been argued in [1], interactions with particles that are *ahead* of the test particle contain only trivial Coulomb repulsion, which has to be subtracted from final expressions in order to get a non-singular result (see the discussion about Coulomb singularity ahead, in this Section). For this reason, in the following we will always assume $\Delta z > 0$.

If we now multiply the retarded electric field $\vec{E}$ generated by a source particle at an observation point $\vec{r_0}(t)$ by the velocity of the test particle $\vec{v_0}$, and by the electron charge e, we get the change of the energy of the test particle due to its interacton with the source particle:

$$\left(\frac{d\mathcal{E}}{dt}\right) = \frac{e^2}{4\pi\varepsilon_0}\left[\frac{c}{\gamma^2}\frac{\vec{n}\cdot\vec{\beta_0} - \vec{\beta}\cdot\vec{\beta_0}}{R^2(1-\vec{n}\cdot\vec{\beta})^3} + \frac{(\vec{n}\cdot\dot{\vec{\beta}})(\vec{n}\cdot\vec{\beta_0} - \vec{\beta}\cdot\vec{\beta_0}) - (\vec{\beta_0}\cdot\dot{\vec{\beta}})(1-\vec{n}\cdot\vec{\beta})}{R(1-\vec{n}\cdot\vec{\beta})^3}\right] . \quad (5)$$

where $\vec{\beta}$ and $\dot{\vec{\beta}}$ are, respectively, the dimensionless velocity and its time derivative at the retarded time t', R is the distance between the retarded position of the source particle and the observation point, and $\hat{n}$ is a unit vector along the line connecting those two points.

The first term on the right side of Eq. (5) is proportional to R^{-2} and is called Coulomb term: as has been argued in [1], it is singular in the limit $R \to 0$ (that is, $\Delta z \to 0$) due to the infinitely small transverse size of the bunch that we use in our model. Following [1], we will cure the situation by subtracting from Eq.(5) its purely Coulomb counterpart corresponding to rectilinear motion of the same two particles with constant velocity:

$$\left(\frac{d\widehat{\mathcal{E}}}{dt}\right) = \left(\frac{d\mathcal{E}}{dt}\right) - \frac{e^2\beta c}{4\pi\varepsilon_0\gamma^2(\Delta z)^2} . \quad (6)$$

The resulting expression appears to be regular in the limit $\Delta z \to 0$. This regularized formula will be used in all following calculations.

In the small-angle approximation, one has to expand the above expressions up to second order terms in the transverse velocity.

Performing this expansion and putting, with the same accuracy, $R \simeq (z_0 - z')$, one gets

$$\left(\frac{d\widehat{\mathcal{E}}}{dt}\right) \simeq \frac{e^2}{4\pi\varepsilon_0}\frac{2\gamma^2}{1+\gamma^2(\vec{n_\perp} - \vec{\beta_\perp}(z'))^2}\{[C] + [R]\} \ , \quad (7)$$

where $\vec{n_\perp}$ is given by

$$\vec{n_\perp} = \frac{1}{(z_0 - z')} \int_{z'}^{z_0} d\zeta \vec{\beta_\perp}(\zeta) \tag{8}$$

and $[C]$ and $[R]$ stand for the Coulomb and the Radiative part, respectively:

$$[C] \equiv \frac{2c}{(z_0 - z')^2}$$
$$\times \left\{ \frac{1 - \gamma^2 \left(\vec{\beta_\perp}(z_0) - \vec{n_\perp}\right)^2 + \gamma^2 \left[\vec{\beta_\perp}(z_0) - \vec{\beta_\perp}(z')\right]^2}{\left[1 + \gamma^2 \left(\vec{n_\perp} - \vec{\beta_\perp}(z')\right)^2\right]^2} \right.$$
$$\left. - \frac{1 + \gamma^2 \left[\vec{n_\perp} - \vec{\beta_\perp}(z')\right]^2}{\left[1 - \gamma^2 \vec{n_\perp}^2 + \gamma^2 (z_0 - z')^{-1} \int_{z'}^{z_0} \vec{\beta_\perp}^2(\zeta) d\zeta\right]^2} \right\}, \tag{9}$$

$$[R] \equiv 2\gamma^2 \frac{\dot{\vec{\beta}}_\perp}{(z_0 - z') \left\{1 + \gamma^2 \left[\vec{n_\perp} - \vec{\beta_\perp}(z')\right]^2\right\}^2}$$
$$\times \left\{ \left[\vec{n_\perp} - \vec{\beta_\perp}(z')\right] \left[1 + \gamma^2 \left(\vec{\beta_\perp}(z_0) - \vec{\beta_\perp}(z')\right)^2 \right.\right.$$
$$\left. - \gamma^2 \left(\vec{n_\perp} - \vec{\beta_\perp}(z_0)\right)^2 \right] - \left[\vec{\beta_\perp}(z_0) - \vec{\beta_\perp}(z')\right]$$
$$\left. \times \left[1 + \gamma^2 \left(\vec{n_\perp} - \vec{\beta_\perp}(z')\right)^2\right] \right\}. \tag{10}$$

A rather straightforward calculation shows that the obtained expression is, indeed, regular in the limit $\Delta z \to 0$ or, equivalently, $(z_0 - z') \to 0$. Namely, it is sufficient to consider the case of constant transverse acceleration $\dot{\vec{\beta}}_\perp = Const$. Without loss of generality, let us put $\dot{\beta}_x = \alpha$, $\dot{\beta}_y = 0$. By shifting the origin and denoting $z_0 - z' \equiv \tau$, one has $\beta_y(z_0) = 0$, $\beta_x(z_0) = \alpha\tau$. Upon this, the Coulomb part becomes

$$[C] \simeq \frac{\gamma^2 \alpha^2}{6} \frac{\left(1 - \frac{1}{3}\gamma^2\alpha^2\tau^2\right)}{\left(1 + \frac{1}{4}\gamma^2\alpha^2\tau^2\right)^2 \left(1 + \frac{1}{12}\gamma^2\alpha^2\tau^2\right)^2}. \tag{11}$$

which clearly has no pole as $\tau \to 0$. Similarly, one can check the absence of singularity in the second term on the right side of Eq. (5) that, being proportional to R^{-1}, is called radiative term.

Next we evaluate the energy change for a test particle interacting with the whole bunch. Under our assumptions it is logical to express the bunch density, that we will call λ, in terms of the longitudinal distance from the test particle. The corresponding variable, Δz, has been already introduced in Eq.(4).

Clearly, in the evaluation of the energy change, it is more convenient to perform integration over the retarded position z' rather than over the distance between particles Δz, since this eliminates the necessity of solving Eq. (3) against z'. After several calculations, the energy change can finally be written as

$$\left(\frac{d\mathcal{E}}{dt}\right)_B (z_0) = \int_{z_0}^{-\infty} \left(\frac{d\widehat{\mathcal{E}}}{dt}\right) (z_0, z') \lambda(\Delta z) \frac{d(\Delta z)}{dz'} dz', \tag{12}$$

where $'B'$ stands for *Bunch* and $\lambda(\Delta z)$ is assumed to behave in such a way that the integral converges. Moreover note that, in Eq. (12), $\lambda(\Delta z)$ is to be considered as a shorthand for $\lambda(\Delta z(z, z'))$ and that

$$\frac{d(\Delta z)}{dz'} = -\left(1 - \hat{\vec{n}} \cdot \vec{\beta}(z')\right). \tag{13}$$

If we want to obtain the energy loss during the entire trajectory we have to integrate over z_0 (or, equivalently, over t) which, finally, gives

$$\Delta\mathcal{E} \simeq \frac{e^2}{4\pi\epsilon_0 c} \int_{-\infty}^{+\infty} dz_0 \int_{-\infty}^{z_0} dz' \{[C] + [R]\} \lambda(\Delta z). \tag{14}$$

where $[C]$ and $[R]$ are defined by Eqs. (9) and (10), and Δz by Eq. (3).

Now, all we need to know in order to evaluate the energy change is the transverse velocity of the bunch as function of the propagation distance. The latter is fully defined by the (pre-designed) configuration of external magnetic fields.

3 COMPUTATIONAL RESULTS

A useful, particular case of Eq. (14) is given when the current profile has rectangular shape: $\lambda(\Delta z)$ is assumed to be constant, $\lambda(\Delta z) = \lambda_0$, over the whole length of the bunch l_b. If the test particle is situated at a distance s_0 from the head of the bunch, then the expression for the energy loss becomes

$$\Delta\mathcal{E}(s_0) \simeq \frac{e^2 \lambda_0}{4\pi\epsilon_0 c} \int_{-\infty}^{+\infty} dz_0 \int_{z'_*(l_b - s_0)}^{z_0} dz' \{[C] + [R]\}. \tag{15}$$

where $z'_*(l_b - s_0)$ stands for the solution of Eq. (3) corresponding to $\Delta z = l_b - s_0$, and s_0 is understood to be positive for particles that lie behind the head of the bunch.

We have performed a comparison of the above expressions with earlier results obtained without the use of the small-angle approximation in [1], where the problem of a 'rigid', one-dimensional (1D) electron bunch entering a circular path from a straight path in vacuum has been carefully studied.

The authors of [1] introduce normalized expressions for the bunch length ($\hat{l}_b = l_b\gamma^3/A$, where A is the radius of curvature of the bend) and for the angular dimension of the magnet ($\hat{\phi}_m = \gamma\phi_m$), and they consider the bunch and the magnet *long* (or *short*) if the corresponding normalized

	$B(T)$	$l_b(m)$	γ	$\overline{z}(m)$	$N/10^9$
1	0.043	$1.0 \cdot 10^{-6}$	25	$1.2 \cdot 10^{-2}$	6.0
2	0.085	$1.0 \cdot 10^{-7}$	50	$8.0 \cdot 10^{-3}$	10.0
3	0.17	0.45	50	$9.9 \cdot 10^{-2}$	10.0
4	0.85	0.2	500	0.02	10.0

Table 1: Parameters chosen for the simulation. B is the magnetic field, in Tesla, l_b is the bunch length in meters, γ is the Lorentz factor, $\overline{z}$ is the length of the interaction zone, in meters and N is the number of particles considered in the bunch.

	$AnalyticalResults(J)$	$SimulationResults(J)$
1	$8.7 \cdot 10^{-15}$	$8.3 \cdot 10^{-15}$
2	$1.54 \cdot 10^{-13}$	$1.50 \cdot 10^{-13}$
3	$3.7 \cdot 10^{-17}$	$3.4 \cdot 10^{-17}$
4	$8.4 \cdot 10^{-17}$	$9.3 \cdot 10^{-17}$

Table 2: Energy change, in Joules, for an electron located at the head of a bunch with rectangular density distribution. A comparison is made between an evaluation done with completely analytical formulas found in [1] and our simulation. Parameters are as in Table 1.

expressions are greater or smaller than unity. Upon this they discuss several limiting cases.

In one of those limiting cases, the comparison is particularly simple: if the bunch is *short* and the bending magnet is, in the normalized sense, *much longer* than the bunch then, with [1], we can assume all the retarded positions of the sources to lie within the bending magnet, and the situation becomes stationary.

For a rectangular bunch containing N particles, one finds upon a calculation similar to that in Eq. (11)

$$\left(\frac{d\mathcal{E}}{dt}\right)_B = -\frac{1}{4\pi\varepsilon_0}\frac{4Ne^2\gamma c}{Al_b}\frac{\gamma u_s\left(8+\gamma^2 u_s^2\right)}{\left(4+u_s^2\gamma^2\right)\left(12+\gamma^2 u_s^2\right)}\,, \tag{16}$$

where

$$u_s \simeq \frac{2\gamma^2(l_b - s_0)}{A}\,. \tag{17}$$

One can easily check that Eqs. (16) and (17) coincide with those found in [1].

In general, the expressions are rather complicated, and the corresponding comparison can only be done numerically. A computer code has been developed and benchmarked against several limiting cases given in [1]. The choice of parameters is presented in Table 1. Results are given in Table 2.

In this table, cases 1 and 2 deal with a *short* bunch and a magnet *longer* than the bunch: here the crucial factor is the energy of the beam. The difference by a factor of 2 in the Lorentz factor is responsible for the increase by a factor of 16 in the energy change. In cases 3 and 4 the magnet is $long$ and the bunch is $much\ longer$ (again in the normalized sense) than the magnet; these two cases have been computed, respectively, with low- and high-energy bunches.

In all cases we observe a good agreement between our numerical computations and the corresponding analytical estimates from [1]. It is also worth mentioning that in all four cases the total energy change is small as compared to the initial particle energy; specifically, the largest relative energy change of about 4% is found in case 2. For these situations, this confirms consistency of the computational scheme a posteriori.

As one more test we calculated, with our method, the instantaneous power radiated by a particle located at the head of a bunch that enters into a bending magnet. This case demonstrates pronounced transient collective effects. To be specific, we considered a 1 mm-long, 40 MeV bunch with rectangular electron density distribution entering a circular trajectory with a radius $A = 1$ m from a straight path. The dependence of the radiated power on the angle of deflection θ is shown in Fig. 2.

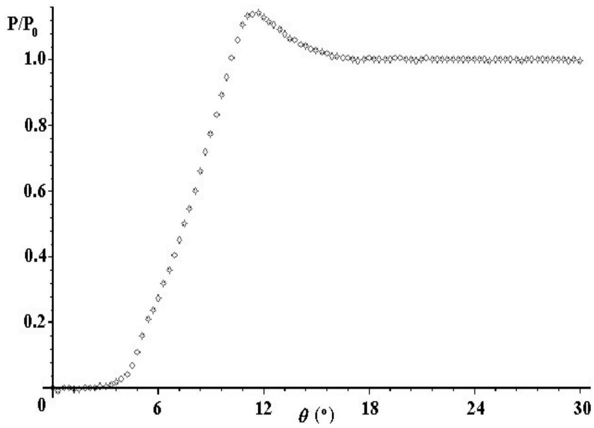

Figure 2: Normalized transient power loss for a bunch with rectangular density distribution going into a bend

The observed dependence is in agreement with well-known results [3], [4].

4 LOW-γ REGION INVESTIGATIONS

Another particular, interesting case of Eq. (14) and of Eq. (12) is given when we consider, as in [1], a rigid, 1D bunch with Gaussian particle density distribution entering a bending magnet after coming from an infinitely long straight section.

The bunch standard deviation will be indicated with $\sigma = 50\ \mu m$, and the total charge will be $q = 1\ nC$. In the actual simulation we will truncate the gaussian beam at $\pm 10\ s/\sigma$, where the distribution is understood to be centered at $s/\sigma = 0$. The magnet has a radius $R = 1.5\ m$.

We used our computer code in order to integrate Eq. (12) at different positions of the test particle within the bunch and at different locations of the bunch within the magnet obtaining, therefore, the instantaneous rate of energy

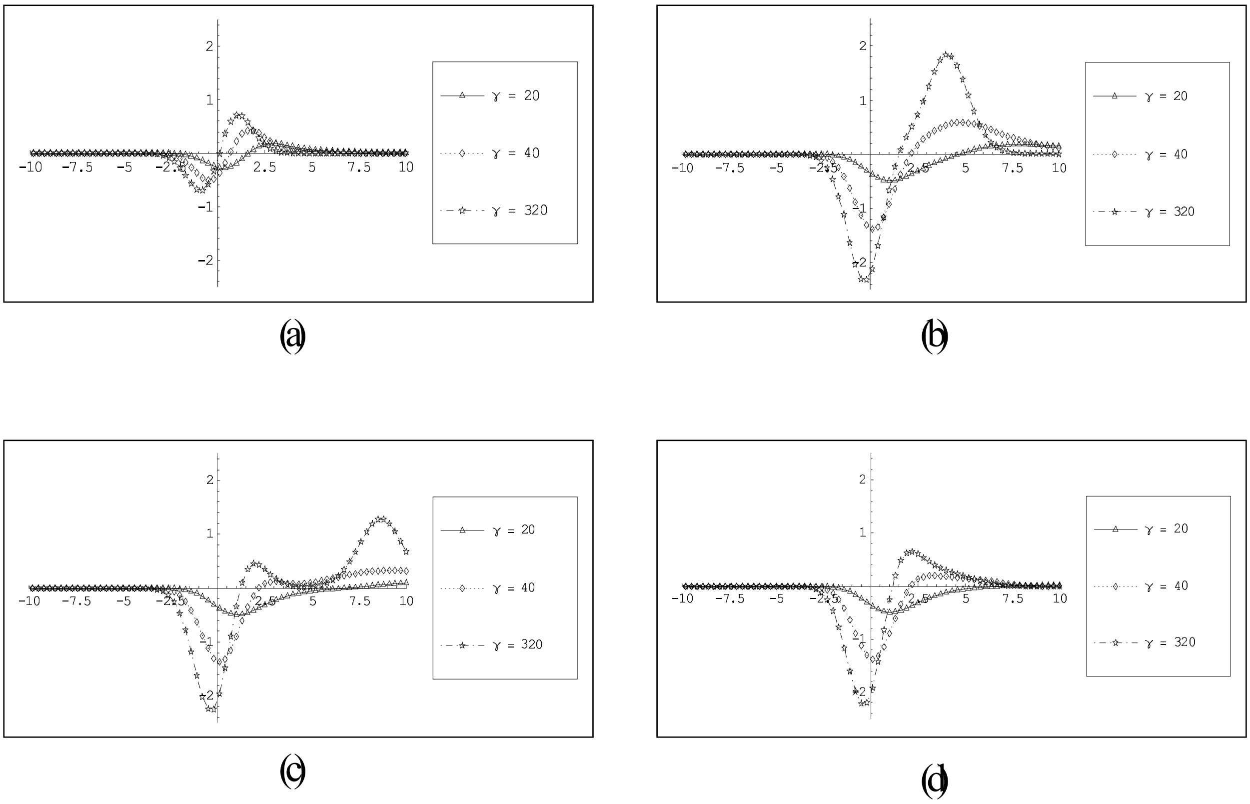

Figure 3: Rate of energy change of an electron in MeV/m as a function of its position along the Gaussian Bunch s/σ entering a magnet, as calculated using our approach. Every picture shows results for different values of γ. Parameters are $R = 1.5\ m$, $\sigma = 50\ \mu m$, $q = 1\ nC$ (a) 5 cm after the entrance. (b) 14 cm after the entrance. (c) 18 cm after the entrance. (d) 25 cm after the entrance.

change of any particle within the bunch for different values of γ.

In Fig. 3 we plotted these rates for several positions of the bunch after the beginning of the magnet. These positions span over the entire transient phenomenon in which we have retarded sources both in the straight line and bending magnet. In every figure we plotted simulation results for different values of γ.

As one can see from Fig. 3, results obtained with our approach exhibit a strong γ-dependence.

Alternatively, in [1], the following approximated formula is used in order to describe the same situation:

$$\frac{d\mathcal{E}}{d(ct)} \simeq -\frac{1}{4\pi\epsilon_0}\frac{2e^2N}{3^{1/3}(2\pi)^{1/2}R^{2/3}\sigma^{4/3}} \times \Bigg[\rho^{-1/3}\left(e^{-(\xi-\rho)^2/2} - e^{-(\xi-4\rho)^2/2}\right) + \int_{\xi-\rho}^{\xi}\frac{d\xi'}{(\xi-\xi')^{1/3}}\frac{d}{d\xi'}e^{-\xi'^2/2}\Bigg], \qquad (18)$$

where $\xi = s/\sigma$, $\rho = R\phi^3/24\sigma$ and ϕ is an angle fixing the position of the bunch inside the magnet (see Fig. 4).

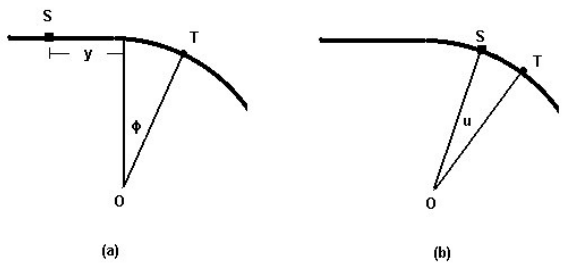

Figure 4: Geometry for the present position of a test particle T and the retarded position of a source particle S. (a) S is in the straight line. (b) S is in the bend.

Eq. (18) is clearly independent of γ, as also Fig. 5, which we took from [1] (integrating Eq. 18) shows.

It is useful to remember the hypothesis under which Eq. (18) was derived. By considering the transient phenomenon *in toto*, that is up to saturation to the steady state

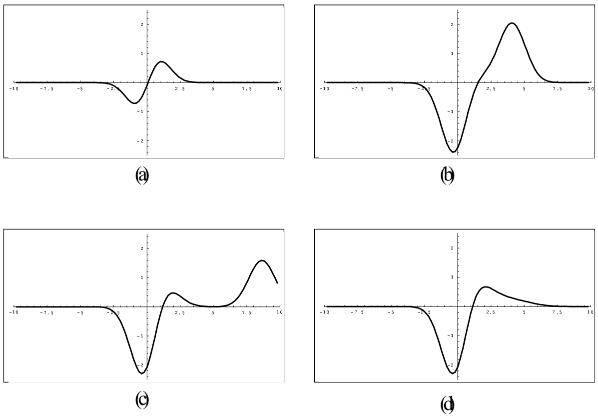

Figure 5: Analytical results from literature ([1]). Rate of energy change of an electron in MeV/m as a function of its position along the Gaussian Bunch s/σ entering a magnet. Parameters are $R = 1.5\ m$, $\sigma = 50\ \mu m$, $q = 1\ nC$. The plots are γ-independent. (a) 5 cm after the entrance. (b) 14 cm after the entrance. (c) 18 cm after the entrance. (d) 25 cm after the entrance.

regime (when all the retarded sources are in the bend) we implicitly make the assumption that the bending magnet is long enough to allow the entire bunch entering such a regime. Besides this assumption, Eq. (18) was derived, in [1], when the following conditions are met:

$$\frac{R}{\gamma^3}\frac{d\lambda(s)}{ds} \ll \lambda(s), \tag{19}$$

and

$$\hat{\phi}_b \gg 1, \tag{20}$$

being

$$\frac{\hat{\phi}_b}{2} + \frac{\hat{\phi}_b^3}{24} = \frac{\gamma^3}{R} l_b, \tag{21}$$

where l_b is the bunch length. Eq.s (19) and (20) fix a lower limit to the values of γ above which Eq. (18) is valid. Let us search for an extension of Eq. (18) that is valid regardless of whether such conditions are met, with the intent of comparing it with our results in Fig. 3.

In order to do so we start with Fig. 4, and with the equations, also derived in [1], for the rate of energy change of a test particle T at some point in the bending due to the interaction with source particles S with retarded position in the straight line before the bend

$$\frac{d\mathcal{E}}{d(ct)} = \frac{2e^2\gamma\lambda_0}{4\pi\epsilon_0 R}\int_0^\infty d\hat{y}\left\{\frac{(\hat{\phi}+\hat{y})^2 + \hat{\phi}^3(3\hat{\phi}/4+\hat{y})}{[(\hat{\phi}+\hat{y})^2+\hat{\phi}^4/4]^2} - \frac{(\hat{\phi}+\hat{y})^2+\hat{\phi}^4/4}{[(\hat{\phi}+\hat{y})^2+(\hat{\phi}^3/12)(\hat{\phi}/4+\hat{y})]^2}\right\} \tag{22}$$

and inside the arc

$$\frac{d\mathcal{E}}{d(ct)} = \frac{2e^2\gamma\lambda_0}{4\pi\epsilon_0 R}\int_0^{\hat{\phi}} d\hat{u}\left(1+\frac{\hat{u}^2}{4}\right)\left\{\frac{\hat{u}^2/4-1}{2(1+\hat{u}^2/4)^3} + \frac{1}{\hat{u}^2}\left[\frac{1+3\hat{u}^2/4}{(1+\hat{u}^2/4)^3} - \frac{1}{(1+\hat{u}^2/12)^2}\right]\right\} \tag{23}$$

that are valid in the case of an infinite long electron bunch with constant particle density distribution λ_0. Here, as in [1] and above in this paper, $\hat{\phi} = \gamma\phi$, $\hat{y} = y\gamma/R$ and $\hat{u} = \gamma u$ being the meaning of y, ϕ and u explained in Fig. 4. Moreover, always in [1], we find the following relations, that hold, respectively, when S is in the straight line

$$(\hat{s}-\hat{s}') = \frac{\hat{\phi}+\hat{y}}{2} + \frac{\hat{\phi}^3}{24}\frac{\hat{\phi}+4\hat{y}}{\hat{\phi}+\hat{y}} \tag{24}$$

or in the bend

$$(\hat{s}-\hat{s}') = \frac{\hat{u}}{2} + \frac{\hat{u}^3}{24}, \tag{25}$$

where $(\hat{s}-\hat{s}') = (s-s')\gamma^3/R$, and $(s-s')$ being the curvilinear distance between the test particle and a source *at the same time*.

From Eq.s (22) to (25), we can write down the following extension of Eq. (18) for the total rate of energy change of an electron entering a bending magnet as a function of its position along the Gaussian bunch:

$$\begin{aligned}\frac{d\mathcal{E}}{d(ct)} = \frac{2e^2\gamma\lambda_0}{4\pi\epsilon_0 R}\Bigg\{ &\int_0^\infty d\hat{y}\left[\frac{(\hat{\phi}+\hat{y})^2+\hat{\phi}^3(3\hat{\phi}/4+\hat{y})}{[(\hat{\phi}+\hat{y})^2+\hat{\phi}^4/4]^2} - \frac{(\hat{\phi}+\hat{y})^2+\hat{\phi}^4/4}{[(\hat{\phi}+\hat{y})^2+(\hat{\phi}^3/12)(\hat{\phi}/4+\hat{y})]^2}\right] \times e^{-\frac{\left\{s-\frac{R}{\gamma^3}\frac{(\hat{\phi}+\hat{y})}{2}+\frac{\hat{\phi}^3}{24}\frac{(\hat{\phi}+4\hat{y})}{(\hat{\phi}+\hat{y})}\right\}^2}{2\sigma^2}} \\ &+ \int_0^{\hat{\phi}} d\hat{u}\left(1+\frac{\hat{u}^2}{4}\right)\left[\frac{\hat{u}^2/4-1}{2(1+\hat{u}^2/4)^3} + \frac{1}{\hat{u}^2}\left(\frac{1+3\hat{u}^2/4}{(1+\hat{u}^2/4)^3} - \frac{1}{(1+\hat{u}^2/12)^2}\right)\times e^{-\frac{\left\{s-\frac{R}{\gamma^3}(\frac{\hat{u}}{2}+\frac{\hat{u}^3}{24})\right\}^2}{2\sigma^2}}\right]\Bigg\}\end{aligned} \tag{26}$$

Next, we integrate numerically Eq. (26) for the same positions of the beam in the magnet and for the same values of γ that we reported in Fig. 3 and in Fig. 5. Results are shown in Fig. 6.

As one can see there is perfect agreement with our simulation results in Fig. 3. Moreover, the γ-dependence vanishes for high values of γ, thus approaching results in [1] (see Fig. 5). This reflects the fact that our general small-angle approximation was successfully applied to the particular case of a transient between a straight line and a bend: in other words, the perfect correspondence between Fig. 3 and Fig. 6 shows that, as expected, Eq. (12), that is valid

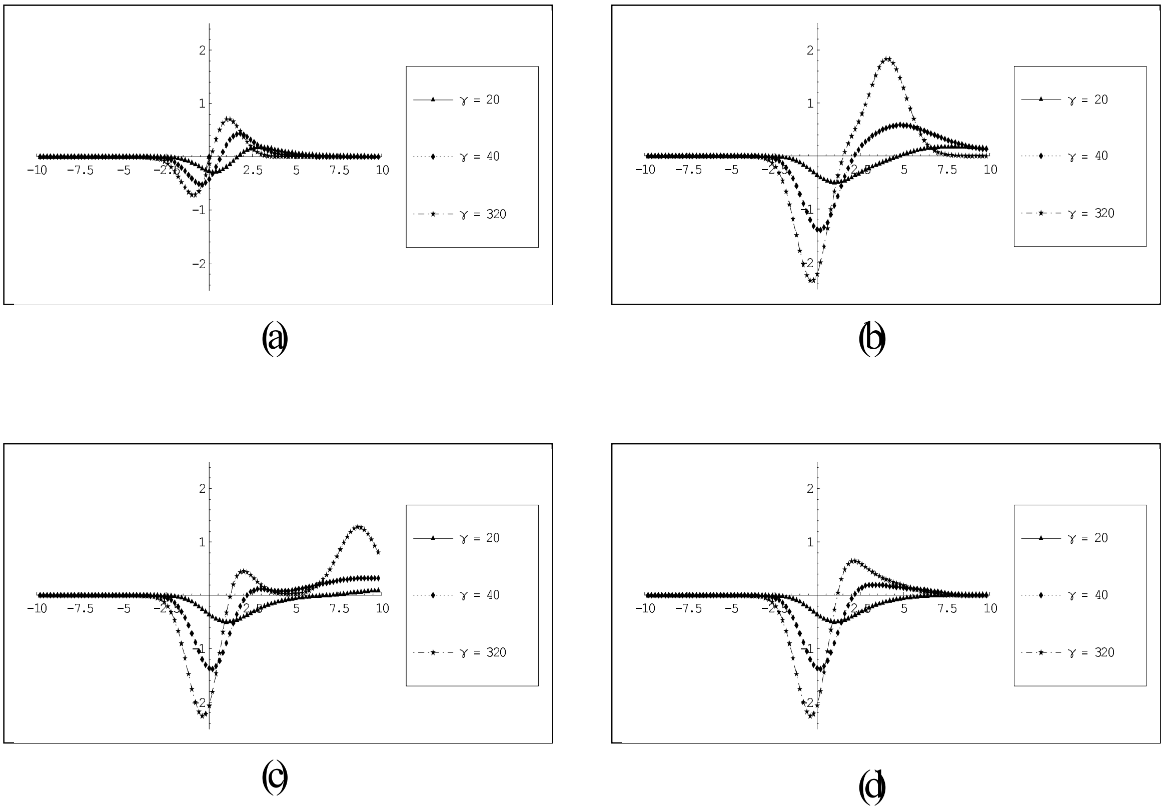

Figure 6: Rate of energy change of an electron in MeV/m as a function of its position s/σ along the Gaussian Bunch entering a magnet, as calculated using a generalization of Eq. (18). Every picture shows results for different values of γ. Parameters are $R = 1.5\ m$, $\sigma = 50\ \mu m$, $q = 1\ nC$ (a) 5 cm after the entrance. (b) 14 cm after the entrance. (c) 18 cm after the entrance. (d) 25 cm after the entrance.

for any trajectory, reduces to Eq. (26) when the appropriate one (straight line followed by a bend) is selected. Of course both reduce to Eq. (18) as soon as the conditions expressed in Eq. (19) and Eq. (20) are satisfied.

5 CONCLUSIONS AND SPECULATIONS

The problem of radiative collective effects within an ultra relativistic electron bunch has been addressed in a new analytical approach. The systematic use of a small-angle approximation results in a new expression for the energy exchange between a test particle and the bunch: all we need to know to evaluate it is the transverse velocity of the bunch as a function of the propagation distance, which is directly determined by the external field configuration.

Analytical and numerical comparisons of the obtained formulas with earlier results by other authors has been performed and a good agreement has been demonstrated.

Moreover, with the help of our method, we studied CSR effects during a transient from a straight line to a circular path. We investigated, in particular, the low-energy case, where the CSR longitudinal force is γ-dependent. This study finally led to an extension of Eq. (18), found in [1], to the energy-dependent region. The perfect agreement between such an extension and our simulation results reflects the fact that, once more, our general approach succeeded in dealing with particular cases. In other words our expression (12) reduces to Eq. (26) when the appropriate trajectory (straight line followed by a bend) is selected.

Our technique is applicable to an arbitrary bunch trajectory subject to the only restriction of a small deviation from the initial direction. We expect our approach to constitute a useful tool in actual magnetic structure design, possibly combined with a particle tracking code in a self-consistent simulation.

6 ACKNOWLEDGEMENTS

Authors would like to acknowledge Bruce Carlsten (Los Alamos National Laboratory), Rui Li and Yaroslav Derbenev (both at Jefferson Lab) and Evgeni Saldin (DESY) for useful discussions. Moreover, G.G. wishes to express his most sincere gratitude to Rui Li for her continuous advice, especially during the last investigations described in this paper. One of us (V.G.) greatly appreciates financial support from CPS (The Dutch Research School on Plasma and Radiation Studies) during his work on this article.

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